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THE USE OF SUBSAMPLE VALUES FOR ESTIMATING A GENERAL STATISTIC FROM INTEGRATED TIME SERIES

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ABSTRACT

This paper demonstrates that the subsampling method of Carlstein (1986) and Fukuchi (1999) works not only for weakly dependent time series but also for time series that are integrated. Specifically, in a multivariate setting, the subsampling method is shown to consistently estimate a general statistic from time series which, or their first differences, are weakly dependent. Moreover, the subsampling method is proved to be consistent for estimating the long-run average relation in regression among integrated, but not cointegrated, time series.

Key Words: Subsampling, L_2-consistency, Weak Dependence, Integration, Spurious Regression
JEL Classifications: C13, C22

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1. INTRODUCTION

The use of subsample values to approximate parameters of the sampling distribution of a statistic was first proposed by Carlstein (1986). Specifically, he showed that the subsampling estimator for the variance of a general statistic is L_2-consistent, provided that the underlying data generating process is stationary and weakly dependent (strong-mixing). Later, Fukuchi (1999) extended the result of Carlstein for a stationary process to a mixing, but possibly nonstationary, process and proved its strong consistency. Fukuchi also showed that, as an immediate consequence, the subsampling estimator for the mean squared error of a predicator is consistent and can be used as a model selection criterion in time series modelling.

In this paper, we extend Fukuchi's result in two ways. First, we generalize the result of Fukuchi for a univariate time series to multivariate time series. Next, rather than requiring the underlying process to be weakly dependent (strong mixing) we show that, for L_2-consistency, imposing mixing condition on the regarding subsample statistic is sufficient. By these extensions, we are able to show consistency of the subsampling estimator even when the regarding multivariate time series are possibly integrated of order one (I(1)) and in turn justify the use of Fukuchi's model selection procedure in such a case.

An interesting and important application of our subsampling result is to the problem of spurious regression. Consider, for example, a regression with two non-cointegrated I(1) series, the OLS estimation for the slope is well known to have a nondegenerate limit distribution and the regression is regarded as spurious since it fails to identify any fixed long-run relation between the two series (Granger and Newbold (1974), Phillips (1986, 1998) and Phillips and Moon (1999)). According to Phillips and Moon (1999), such a failure is due to the strong noise in the regression residuals and because the noise may be attenuated by pooling a large number of cross section and time series units – as long as the noise is independent across the cross section units – they suggested solving the problem via a panel-pooled regression. By contrast, we suggest a solution by means of subsampling. In particular, we show that the long-run average relation of Phillips and Moon (1999) can be consistently estimated through the subsampling method.

The paper is organized as follows. Section 2 provides L_2-consistency for the subsampling estimator in multivariate time series, weakly dependent or integrated. Section 3 presents a consistent subsampling estimator for the long-run relation among non-cointegrated I(1) series. Section 4 concludes. All the proofs are given in the appendix.

2. SUBSAMPLING WITH MULTIVARIATE TIME SERIES

Let $\{Z_t\}_{t\geq 1}$ be a sequence of random n-vetors and let $Z^{(T)} = (Z_1, Z_2, ..., Z_T)$ be a sample of it. Let F_p^q be the σ-field generated by $\{Z_t\}_{t\geq p}^q$. Denote

$$\alpha(m) = \sup_j \sup_{A\in F_{-\infty}^j,\, B\in F_{j+m}^\infty} \left| P(A\cap B) - P(A)P(B) \right|, \qquad (1)$$

and define strong-mixing (α-mixing) to mean $\lim_{m\to\infty}\alpha(m) = 0$. The subsampling method is illustrated as follows. Suppose we would like to estimate $\Theta_T = E[f_T(Z^{(T)})]$ where $f_T : R^{n\times T} \to R^l$ with $l \geq 1$ is a measurable function. Denote $Z_i^{(k)} = (Z_i, Z_{i+2}, ..., Z_{i+k-1})$, a subsample of $Z^{(T)}$, and let $f_{i,k} = f_k(Z_i^{(k)})$. The subsampling estimator of Θ_T with subsample size b is defined as

$$\hat{\Theta}_{T,b} = \frac{1}{T^*}\sum_{i=1}^{T^*} f_{i,b}, \qquad (2)$$

where $T^* = T - b + 1$. Note that the subsample size b is a function of T, b_T, but for notational simplicity, the subscript T is suppressed. Throughout the paper, "$b \to \infty$ and $b/T \to 0$ as $T \to \infty$" is assumed. Also, denote $\hat{\Theta}_T$ ($= \hat{\Theta}_{T,T} = \hat{f}_{1,T}$) as the corresponding full-sample estimator.

To show consistency for $\hat{\Theta}_{T,b}$, we need:

ASSUMPTION 1.

(a) There exists $\Theta \in R^l$ such that (i) $\hat{\Theta}_T \to_p \Theta$ as $T \to \infty$, or $\lim_{T\to\infty} E(\hat{\Theta}_T) = \Theta$, (ii) $(T^*)^{-1}\sum_{i=1}^{T^*} E(f_{i,b}) \to \Theta$ as $T \to \infty$;

(b) $\{(f_{i,b})^2,\ i = 1, ..., T^*\}$ is uniformly integrable;

(c) $\{f_{i,b},\ i = 1, 2, ..., T^*\}$ is strong mixing.

Assumption 1 resembles the assumptions in Fukuchi (1999) except for a few differences. First, of course, Assumption 1 is made in a multivariate setting ($n \geq 1,\ l \geq 1$) and Assumption 1 of Fukuchi is a special case of it ($n = 1,\ l = 1$). Second, part (a) of this assumption requires the full-sample estimator $\hat{\Theta}_T$ is either consistent ("$\to_p$" means convergent in probability) or asymptotically unbiased – in contrast, Fukuchi assumes that $\hat{\Theta}_T$ has to be consistent. The importance of allowing asymptotically unbiased (but possibly inconsistent) for $\hat{\Theta}_T$ will become clear when the result is applied to spurious regression in the next section. Third, instead of assuming that the underlying process Z_t is strong mixing as in Fukuchi (1999), in Assumption 1(c) the strong mixing condition is imposed on the subsample statistics f_b^i. Note that this assumption is weaker than that of Fukuchi, and the same assumption has been used to construct subsampling confidence intervals in autoregressive models that possibly contain a unit root in Politis et al (1999) and Romano and Wolf (2001). Below, we will display that Assumption 1(c) holds if Z_t or the difference of it is strong mixing.

Denote the mixing coefficient corresponding to $\{f_{i,b}\}_{i\geq 1}$ by $\alpha^*(\cdot)$ and let G_p^q be the σ-field generated by $\{f_{i,b}\}_{i\geq p}^q$. Since $f_{i,b}$ is a measurable on any σ-field on which each of $Z_i, Z_{i+1}, ..., Z_{i+b-1}$ is measurable, we have $G_1^j \subseteq F_1^{j+b-1}$ and $G_{j+m}^\infty \subseteq F_{j+m}^\infty$. Consequently, $\alpha^*(m+b-1) \leq \alpha(m)$ (or equally, $\alpha^*(m) \leq \alpha(m-b+1)$) for $m \geq b+1$. Given that $\{Z_t\}_{t\geq 1}$ is strong mixing, "$\alpha(m) \to 0$ as $m \to \infty$", and together with "$b \to \infty$ and $b/T \to 0$ as $T \to \infty$", if Z_t is strong mixing then $f_{i,b}$ is too.[1]

Also, with appropriate demeaning, Assumption 1(c) holds even if the difference of Z_t is weakly dependent (i.e. Z_t is integrated of order one). Suppose Z_t be an n-vector integrated process: $Z_t = Z_{t-1} + U_t$, t=1,2,…,T, with Z_0 being a constant or a random variable with a certain specified distribution.[2] Assume that U_t is strong mixing. Consider the i$^{\text{th}}$ mean-adjusted subsample (of size b): $\dot{Z}_i^{(b)} = (Z_i - \bar{Z}_{i,b}, Z_{i+1} - \bar{Z}_{i,b}, ..., Z_{i+b-1} - \bar{Z}_{i,b})$ where $\bar{Z}_{i,b}$ is the subsample mean: $\bar{Z}_{i,b} = b^{-1}\sum_{h=1}^{b} Z_{i+h-1}$. Define the subsample statistics $\dot{f}_{i,b} = f_b(\dot{Z}_i^{(b)})$. Because $Z_{i+k} - \bar{Z}_{i,b} = \sum_{h=1}^{k} U_{i+h-1} - b^{-1}(\sum_{h=1}^{b} U_{i+h-1})$, for k=1,2, …, b, $\dot{f}_{i,b}$ is a function of $(U_i, U_{i+1}, ..., U_{i+b-1})$ only. Now, given that U_t is strong mixing, $\dot{f}_{i,b}$ is too. Clearly, such a mean-adjusted scheme is crucial for the argument; however, our argument should hold if the targeting statistic is obtained from a linear model with intercept. If the data is not demeaned or if the regarding models does not have an intercept, because of the initial effect, $\dot{f}_{i,b}$ will be a function of Z_i and the subsample statistics series can hardly be weakly dependent. See also Romano and Wolf (2001) and Choi (2005) for related discussions.

Theorem 1(a) of Fukuchi (1999) can be extended as follows.

THEOREM 1. Suppose Assumption 1 holds. It follows that $\hat{\Theta}_{T,b} \to_{L_2} \Theta$ when $T \to \infty$, for $b \to \infty$ and $b/T \to 0$ as $T \to \infty$. (Denote "$\to_{L_2}$" as L_2-consistence.)

A few remarks concerning Theorem 1 are as follows.

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[1] But, since b is not finite as $T \to \infty$, the two processes might have different mixing orders.

[2] See, for example, Phillips (1986).

REMARK 1. Theorem 1 gives a non-trivial extension of Theorem 1(a) in Fukuchi (1999). With this extension, we are able to claim that the subsampling estimator is consistent even if its underlying multivariate time-series processes are I(1)..

REMARK 2. Based on Theorem 1 and with a few additional required assumptions, Theorem 2(a) in Fukuchi (1999) can be extended to show L_2-consistency for the subsampling estimation of the risk of prediction, provided that the series of the subsample predictors is strong mixing. Accordingly, the subsampling model selection criterion of Fukuchi should work in multivariate time series, stationary or integrated.

REMARK 3. With some extra appropriate assumptions, we conjecture that strong consistency for $\hat{\Theta}_{T,b}$ should hold too.

3. SPURIOUS REGRESSION

Let $Z_t' = (y_t, X_t)$ where y_t (a scalar) and X_t (a k-vector) are integrated processes of order one: $y_t = y_{t-1} + v_t$, $X_t = X_{t-1} + w_t$, t=1,2,…T, with y_0 and X_0 being a constant or a random variable with a certain specified distribution. Let $\xi_t' = (v_t, w_t)$. Following Phillips (1986), we make the subsequent assumption.

ASSUMPTION 2.

(a) $E(\xi_t) = 0$ for all t;

(b) $\sup_{i,t}(E\,|\,\xi_{it}\,|^{\beta+\varepsilon}) \leq \infty$ for some $\beta > 2$ and $\varepsilon > 0$;

(c) $\{\xi_t\}_1^\infty$ is strong mixing with mixing parameter $\alpha(m)$ satisfying $\sum_1^\infty \alpha(m)^{1-2/\beta} < \infty$.

(d) $\Sigma = \lim_{T\to\infty} T^{-1} E(Z_T Z_T')$ exists and is positive definite.

With Assumption 2(d), Σ is non-singular and y_t and X_t can not be cointegrated. Under Assumption 2, the multivariate invariance principle gives

$$T^{-1/2} Z_{[rT]} \Rightarrow \Sigma^{1/2} W(r). \tag{3}$$

Here, $\Sigma^{1/2}$ is the symmetric positive definite square root of Σ, $\Rightarrow$ denotes weak convergence, $[rT]$ is the integer part of rT and $r \in (0,1]$, and W(r) is a vector Weiner process composed of k+1 mutually independent univariate Weiner processes.

Consider the regression: $y_t = \alpha + \beta' X_t + \varepsilon_t$. Let $\hat{\beta}$ be the OLS estimator of β. According to Phillips (Theorem 2, 1986):

$$\hat{\beta} \Rightarrow A_{22}^{-1} a_{21}, \tag{4}$$

where

$$A = \begin{bmatrix} a_{11} & a_{21}' \\ a_{21} & A_{22} \end{bmatrix} = \Sigma^{1/2} \left[\int_0^1 \left(W(r) - \int_0^1 W(s) ds \right) \left(W(r) - \int_0^1 W(s) ds \right)' dr \right] \Sigma^{1/2}. \tag{5}$$

Note that a_{11} is a scalar, and a_{21} ($k \times 1$), A_{22} ($k \times k$) are two submatrices, and A is positive definite. Since $\hat{\beta}$ has a nondegenerate limit distribution, the OLS estimator is inconsistent and fails to identity any relation (including no relation) between y_t and X_t. In the literature, a regression of this sort is characterized as spurious (Granger and Newbold (1974), Phillips (1986)).

Recently, Phillips and Moon (1999) proposed using

$$\beta^* = E(A_{22})^{-1} E(a_{21}) \tag{6}$$

to define the long-run average relation between y_t and X_t. As a special case, if y_t and X_t are independent to each other (so the long run relation does not exist), since $E(a_{21}) = 0$, $\beta^* = 0$. According to Phillips and Moon, the failure of OLS is due to the strong noise in the regression residuals and because pooling large time series and cross section components may calm the noise, they suggested a panel-pooled estimator for β^*. Specifically, as shown by Phillips and Moon, the panel-pooled estimator is consistent under cross section independence. By contrast, in what follows, we suggest a subsampling approach.

Let $h_{i,b} = b^{-2} \sum_{t=i}^{i+b-1} (y_t - \bar{y}_{i,b})(X_t - \bar{X}_{i,b})'$ and $g_{i,b} = b^{-2} \sum_{t=i}^{i+b-1} (X_t - \bar{X}_{i,b})(X_t - \bar{X}_{i,b})'$ where $\bar{y}_{i,b} = b^{-1} \sum_{h=1}^{b} y_{i+h-1}$ and $\bar{X}_{i,b} = b^{-1} \sum_{h=1}^{b} X_{i+h-1}$ are the i^th^ subsample means of y_t and X_t, respectively. Denote the full-sample counterparts of $h_{i,b}$ and $g_{i,b}$: $h_T = h_{1,T}$

and $h_T = h_{i,T}$. Also, let $\tilde{H}_{T,b} = (T^*)^{-1}\sum_{i=1}^{T^*} h_{i,b}$ and $\tilde{G}_{T,b} = (T^*)^{-1}\sum_{i=1}^{T^*} g_{i,b}$, $T^* = T + b - 1$, and define subsampling estimator for β^* by

$$\tilde{\beta}_{T,b} = \left(\tilde{G}_{T,b}\right)^{-1} \tilde{H}_{T,b}$$

$$= \left(\sum_{i=1}^{T^*} g_{i,b}\right)^{-1} \left(\sum_{i=1}^{T^*} h_{i,b}\right)$$

$$= \left[\sum_{i=1}^{T^*} \sum_{t=i}^{i+b-1} \left(X_t - \bar{X}_{i,b}\right)\left(X_t - \bar{X}_{i,b}\right)'\right]^{-1} \left[\sum_{i=1}^{T^*} \sum_{t=i}^{i+b-1} \left(y_t - \bar{y}_{i,b}\right)\left(X_t - \bar{X}_{i,b}\right)'\right]. \quad (7)$$

Of course, if we show that $\tilde{H}_{T,b}$ and $\tilde{G}_{T,b}$ are consistent estimators for $E(a_{21})$ and $E(A_{22})$, respectively, then consistency of $\tilde{\beta}_{T,b}$ follows immediately.

THEOREM 2. Suppose Assumption 2 holds. It follows that $\tilde{H}_{T,b} \to_{L_2} E(a_{21})$, $\tilde{G}_{T,b} \to_{L_2} E(a_{11})$ when $T \to \infty$ and $b \to \infty$ and $b/T \to 0$ as $T \to \infty$. As a result, $\tilde{\beta}_{T,b} \to_{L_2} \beta^*$.

Several remarks concerning Theorem 2 are in order.

REMARK 1. The main result of Theorem 2 is to show that the subsampling estimator is L_2-consistent for the long-run average relation among integrated (but not cointegrated) processes and this result is obtained without relying on the use of panel data as in Phillips and Moon (1999).

REMARK 2. It can be noted that the full-sample estimators $\tilde{H}_{T,T}$ and $\tilde{G}_{T,T}$ are asymptotic unbiased (but not consistent) for $E(a_{21})$ and $E(a_{22})$, respectively. Interestingly, even though "$(\tilde{G}_{T,T})^{-1}\tilde{H}_{T,T}$" is neither consistent nor asymptotically unbiased for "$E(a_{22})^{-1}E(a_{21})$" $\tilde{\beta}_{T,b}$ is still consistent for "$E(a_{22})^{-1}E(a_{21})$".

REMARK 3. Instead of using $(T^*)^{-1}\sum_{i=1}^{T^*} g_{i,b}^{-1} h_{i,b}$ as the long-run average relation estimator, we use $(\sum_{i=1}^{T^*} g_{i,b})^{-1}(\sum_{i=1}^{T^*} h_{i,b})$. As matter of fact, it can be shown that $(T^*)^{-1}\sum_{i=1}^{T^*} g_{i,b}^{-1} h_{i,b}$ is consistent for $E(a_{22}^{-1} a_{21})$ which, in general, is different from $E(a_{22})^{-1}E(a_{21})$.

REMARK 4. We may adopt the assumptions made in Phillips and Durlauf (1986) to allow cointegration. In that case, OLS is consistent and, by a modification of the proof of Theorem 2, $\tilde{\beta}_{T,b}$ can be shown too.

Finally, a simple Monte Carlo experiment is conducted to study the finite sample performance of our subsampling estimator. We generate a bivariate I(1) process: $Z_t = Z_{t-1} + \zeta_t$ for t=1,2,…,T, with $Z_0 = 0$ and $\zeta_t \sim_{iid} N(0, I_2)$. We set the subsample size b the integer part of $T^{1/2}$. The results are calculated using 10,000 iterations at different sample sizes and summarized in Table 1. Evidently, the subsampling estimator is converging to zero (the variance is getting smaller) as the sample size increases while OLS is not (the variance is more or less a constant regardless the sample size). Besides, the subsampling estimator appears to outperform OLS even when the sample size is as small as T=50.

Table 1

	OLS		Subsampling	
T	mean	variance	mean	variance
50	-0.0046	0.4036	0.0003	0.0474
100	0.0055	0.3852	0.0006	0.0319
500	-0.0009	0.4027	-0.0007	0.0127
1000	0.0019	0.4031	0.0009	0.0093
5000	0.0101	0.3990	0.0004	0.0042
10000	0.0059	0.3898	-0.0001	0.0028

4. CONCLUSION

This paper extends the work of Fukuchi (1999) to a multivariate setting. L_2-consistency for the subsampling estimator is established when the underlying time-series processes or their differences are weakly dependent. It is shown that the long-run average relation among integrated processes, but not cointegrated, can be consistently estimated through subsampling. In the future, given the fact that spurious regression can also occur for fractionally-integrated processes (Tsay and Chung (2000)) it is of interest to see if the subsampling method works in such a context.

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APPENDIX

Appendix I (Proof of Theorem 1)

Let $\hat{\Theta}_{T,b} = (\hat{\Theta}^1_{T,b}, \hat{\Theta}^2_{T,b}, ..., \hat{\Theta}^l_{T,b})$. By Assumption 1(a), it suffices to show

$$\lim_{T\to\infty} \text{var}(\hat{\Theta}^k_{T,b}) = 0 \quad \text{(A1)}$$

for any k=1,2,…,*l*. Adopt the proof of Theorem 1(a) in Fukuchi (1999), let $v^b_{i,j}$ $= \text{cov}(f_{i,b}, f_{j,b})$, we have

$$(T^*)^2 \text{ var}(\hat{\Theta}^k_{T,b}) \le \sum_{i=1}^{T^*}\sum_{j=1}^{T^*}\left|v^b_{i,j}\right| = A_1 + A_2 + A_3 + A_4 + A_5, \quad \text{(A2)}$$

where $T^* = T - b + 1$, $A_1 = \sum_{i=1}^{b}\sum_{j=1}^{T^*}|v^b_{i,j}|$, $A_2 = \sum_{i=b+1}^{T^*-b}\sum_{j=1}^{i-b}|v^b_{i,j}|$, $A_3 = \sum_{i=b+1}^{T^*-b}\sum_{j=i-b+1}^{i+b}|v^b_{i,j}|$, $A_4 = \sum_{i=b+1}^{T^*-b}\sum_{j=i+b+1}^{T^*}|v^b_{i,j}|$ and $A_5 = \sum_{i=T^*-b+1}^{T^*}\sum_{j=1}^{T^*}|v^b_{i,j}|$.

First, By Assumption 1(b), it follows that A_1, A_3 and A_5 are all $O(bT)$. Therefore, $\lim_{T\to\infty}(T^*)^{-2}(A_1 + A_3 + A_5) = 0$.

Next, for any $\Delta \ge 0$, define ${}_{\Delta}X = X\cdot I\{|X| < \Delta\}$ and ${}^{\Delta}X = X - {}_{\Delta}X$. Also, let $C_{i,j}$ $= \max\{E[(f_{i,b})^2], E[(f_{j,b})^2]\}$ and ${}^{\Delta}C_{i,j} = \max\{E[({}^{\Delta}f_{i,b})^2], E[({}^{\Delta}f_{j,b})^2]\}$. Note $\alpha^*(\cdot)$ is the mixing coefficient of $\{f_{i,b}\}_{i\ge1}$. By Assumption 1(b),(c) and applying Lemma 1 of Carlstein (1986), for any $\Delta > 0$, we have

$$(T^*)^{-2}A_2 \le (T^*)^{-2}\sum_{i=b+1}^{T^*-b}\sum_{j=1}^{i-b}\left[4\Delta^2\alpha^*(i-j) + 3C^{1/2}_{i,j}\,{}^{\Delta}C^{1/2}_{i,j}\right]. \quad \text{(A3)}$$

Since $\alpha^*(b+i) \to 0$ as $b \to \infty$, $C_{i,j}$ is finite, and ${}^{\Delta}C_{i,j}$ goes to zero as $\Delta \to \infty$ almost surely for any i and j (by Assumption 1(b): $(f_{i,b})^2$ is uniformly integrable), we have $\lim_{\Delta\to\infty}\limsup_{T\to\infty}[(T^*)^{-2}A_2] = 0$. By the same argument, it can be shown that $\lim_{\Delta\to\infty}\limsup_{T\to\infty}[(T^*)^{-2}A_4] = 0$. We thus complete the proof.♦

Appendix II (Proof of Theorem 2)

Let $Z_t' = (y_t, X_t)$ and let $\bar{Z}$ be its sample mean. Under Assumption 2, we have, as $T \to \infty$,

$$A_T = T^{-2} \sum_{t=1}^{T} \left(Z_t - \bar{Z}\right)\left(Z_t - \bar{Z}\right)'$$

$$\Rightarrow A = \Sigma^{1/2} \left[\int_0^1 \left(W(r) - \int_0^1 W(s)ds \right) \left(W(r) - \int_0^1 W(s)ds \right)' dr \right] \Sigma^{1/2}. \qquad \text{(A4)}$$

Also, let $\bar{Z}_{i,b}$ be the sample mean of the i^th^ subsample: $\bar{Z}_{i,b} = b^{-1} \sum_{h=1}^{b} Z_{i+h-1}$. For each i=1,…,$T^*$, under Assumption 2, we have

$$A_{i,b} = b^{-2} \sum_{t=i}^{i+b-1} \left(Z_t - \bar{Z}_{i,b}\right)\left(Z_t - \bar{Z}_{i,b}\right)' \Rightarrow A \qquad \text{(A5)}$$

as $b \to \infty$. Now, since h_T, g_T are components of A_T, and $h_{i,b}$, $g_{i,b}$ are components of $A_{i,b}$, if we show that A_T and $A_{i,b}$ satisfy Assumption 1, then by applying Theorem 1, we complete the proof.

First, by (A4) and (A5), we have $E(A_T) \to E(A)$ and $T^{*-1} \sum_{i=1}^{T^*} E(A_{i,b}) \to E(A)$ as $T \to \infty$; therefore, Assumption 1(a) holds. Now, define $\|\cdot\|$ the Euclidean norm and note by continuous mapping theorem, we have $\|A_{i,b}\|^2 \Rightarrow \|A\|^2$ and

$$E\|A_{i,b}\|^2 = tr(E(vec(A_{i,b})vec(A_{i,b})')) \to tr(E(vec(A)vec(A)')) = E\|A\|^2 < \infty \qquad \text{(A6)}$$

as $b \to \infty$. Consequently, $\|A_{i,b}\|^2$ is uniformly integrable in b and Assumption 1(b) is satisfied. Finally, following the same argument of Section 2, by Assumption 2(c), it follows that $vech(A_{i,b})$ is strong mixing and thus Assumption 1(c) holds.♦

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