Counting the spanning trees of the 3-cube using edge slides

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2009 New Zealand Mathematics Colloquium
1 Introduction
   - Cubes and spanning trees
   - Counting spanning trees: ways and means

2 The 3-cube
   - Edge slides
   - Counting the trees

3 Higher dimensions
The $n$-cube is the graph $Q_n$ with:

- vertices the subsets of $[n] = \{1, 2, \ldots, n\}$;
- an edge between $S$ and $R$ if they differ by adding or deleting a single element.

\[
\begin{align*}
\emptyset & \quad \{1\} & \quad \{2\} & \quad \{3\} \\
\{1\} & \quad \{1, 2\} & \quad \{1, 3\} & \quad \{2, 3\} \\
\{2\} & \quad \{1, 3\} & \quad \{2, 3\} & \\
\{3\} & \quad \{2, 3\} & & \\
\end{align*}
\]
Spanning trees

Definition

A *spanning tree* of a connected graph $G$ is

- a maximal subset of the edges that contains no cycle;
- equivalently,
- a minimal subset of the edges that connects all the vertices.
Counting trees — the matrix way

Theorem (Kirchoff’s Matrix-Tree Theorem)

The number of spanning trees of a simple connected graph $G$ is given by the determinant of a matrix associated with $G$ — the Laplacian of $G$, with row $i$, column $i$ deleted.

$$
\begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 2 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}
$$

1

2

3

4
Theorem (Kirchoff’s Matrix-Tree Theorem)

The number of spanning trees of a simple connected graph $G$ is given by the determinant of a matrix associated with $G$ — the Laplacian of $G$, with row $i$, column $i$ deleted.

$$\begin{vmatrix}
2 & -1 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 1 \\
\end{vmatrix} = 3$$

Counting the spanning trees of the 3-cube

Christopher Tuffley (Massey University)
Counting trees — the combinatorial way

Model: Prüfer code for spanning trees of $K_n$ (Prüfer, 1918)

The *Prüfer code* is a bijection

$$\text{spanning trees of } K_n \leftrightarrow \{1, \ldots, n\}^{n-2}$$

— recovering Cayley’s Theorem that $K_n$ has $n^{n-2}$ spanning trees.

Prüfer code 3411

Counting spanning trees: ways and means
Spanning trees of the $n$-cube

**Known result**

The $n$-cube has

\[
\prod_{\substack{S \subseteq [n] \\ |S| \geq 2}} 2|S| = 2^{2^n-n-1} \prod_{k=1}^{n} \binom{n}{k}
\]

spanning trees.

For $n = 3$ this gives $2^4 \cdot 2^3 \cdot 3 = 384$ spanning trees.

**Proof.**

The Matrix-Tree Theorem + clever determination of eigenvalues.
See e.g. Stanley, *Enumerative Combinatorics*, Vol II.
Spanning trees of the $n$-cube

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**Problem**

Stanley: “A direct combinatorial proof of this formula is not known.”
A weighted count

Theorem (Martin and Reiner, 2003)

With respect to certain weights $q_1, \ldots, q_n, x_1, \ldots, x_n$ we have

$$
\sum_{\text{s. trees of } Q_n} q^\text{dir}(T) x^\text{dd}(T) = q_1 \cdots q_n \prod_{S \subseteq [n]} \sum_{|S| \geq 2} q_i (x_i^{-1} + x_i).
$$

- degree of $q_i$ in $q^\text{dir}(T)$ is the number of edges in direction $i$
- degree of $x_i$ in $x^\text{dd}(T)$ is the number of edges in the “upper” $i$-face minus the number in the “lower”.

Suggests that

a spanning tree of $Q_n$

\[ \uparrow \]

a choice of element and sign at each vertex of cardinality 2.
A weighted count

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Edge slides

**Definition**
An edge of a spanning tree is *slidable* if it can be “slid” across a face of the cube to give a second spanning tree.

**Observation**
An edge that may be slid in direction \( i \) must lie on the path joining two \( i \)-edges.
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The 3-cube

Existence

Lemma
A minimal path joining two $i$-edges contains a unique edge that may be slid in direction $i$.

Proof (length three case only).
Vertices $u$ and $v$ must meet edges of the tree. There are three possibilities.

Corollary
A tree with $k$ edges in direction $i$ has $k - 1$ edges that may be slid in direction $i$, for a total of exactly four possible slides.
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Independency

Lemma

“Parallel” edge slides on the same tree are independent.

Proof.

Our existence proof above was purely local.
Orientations

- Root each spanning tree at $\emptyset$.
- Orient each edge towards the root.
- Let $u_i$ be the number of “upward” edges in direction $i$.

**Lemma**

1. The effect of an $i$-slide on $(u_1, u_2, u_3)$ is to change $u_i$ by $\pm 1$. The sign is determined by the direction of slide.
2. There is a “downward” $i$-slide $\iff u_i > 0$. 
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2. There is a “downward” $i$-slide $\Leftrightarrow u_i > 0$. 
**Upright trees**

**Definition**

A spanning tree is *upright* if it has only “downward” edges.

Given a spanning tree $T$, carry out all possible *downward* steps:
1. 3-slides; then
2. 2-slides; then
3. 1-slides.

The result is an upright tree canonically associated with $T$. 
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Theorem

There are

1. $2^3 \cdot 3$ upright trees, and
2. $2^4$ trees associated with each, for a total of $2^4 \cdot 2^3 \cdot 3 = 384$ trees.
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Decide in turn whether to carry out each

1. 1-slide;
2. 2-slide;
3. 3-slide

— a total of four yes-no decisions.
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There are

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The count can be made bijective with additional attention to orientation.
Can we carry out a similar programme in higher dimensions?
Higher dimensions?

**Question**

*Can we carry out a similar programme in higher dimensions?*

Not yet...

- every tree can be made upright using downward slides,
- but choices are required:
  - a tree may have “extra” edge slides;
  - parallel slides need not be independent
    (perhaps only for $n \geq 5$?).

New ideas are needed to make these choices systematically.