

Defensive Spiral Emergence in a Predator-Prey Model

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Abstract

We report on emergent spiral patterns generated from a predator-prey rule-based automaton model. We discuss some systematic analysis techniques applied to the patterns and hypothesise that their origin lies in a defensive behaviour that arises from the predator-prey interactions. We believe the emergent patterns from this simple model correspond to similar behaviours in real-world situations where two groups interact across opposing fronts such as military formations.

1 Introduction

Inspired by the plethora of interesting behaviours reported for simulated artificial life (ALife) models (Adami, 1998; Ray, 1991; Levy, 1992; Komosiński and Ulatowski, 1999) we were attempting to design an automaton-based (Wolfram, 2002) ALife simulator of our own. We were constructing a framework for animal automata (or “animats” (Meyer and Wilson, 1990)) to evolve using a genotype based around some simulated world-relevant instructions, but were testing the model framework with some simple animal strategies. In the course of this work (James et al., 2004a) we accidentally discovered the spiral pattern formation (shown in figure 1) that arises from a relatively simple predator-prey interaction between randomly initialised groups of animal automata.

We believe we are justified in describing this behaviour as “emergent” according to discussions of the term in the literature (Cariani, 1991; Ronald et al., 1999a; Ronald et al., 1999b). The model we describe in section 2 does not anticipate the spiral pattern generation nor the “battlefronts” and “flanking behaviour” of our predator-prey animats. It requires a different “language” to describe the patterns and behaviour from that of the model and this is one of the criteria for “emergence” noted in (Ronald et al., 1999a).

The model described runs on a desktop computer in manageable times of approximately 5 seconds of simulation time per animal-update time-step (for less than 3000 animals in the

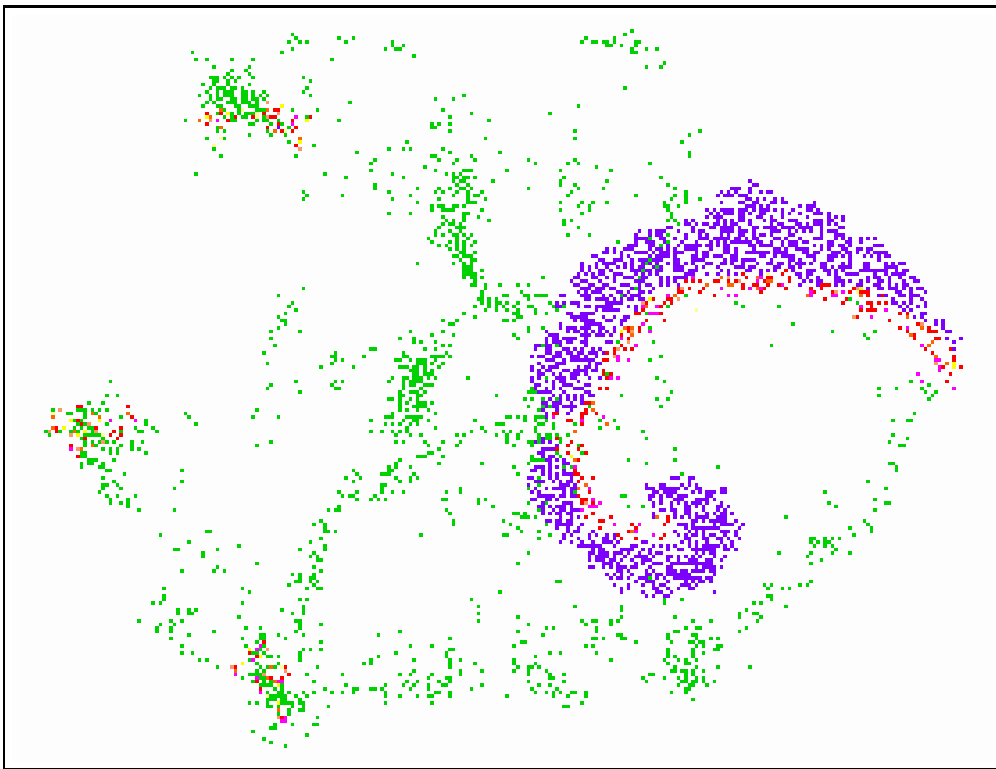


Figure 1. *Visualisation of the predator-prey model state. Dark pixels (blue) are the prey (“rabbits”). Lighter pixels (red) represent predators (“foxes”). The rabbits are relying on safety in numbers while fleeing from the attacking foxes. Very light pixels (green) show the location of fox corpses when they died of starvation. The components are decomposed and discussed further in figure 3.*

systems reported here). We have described elsewhere the multi-phasic simulation techniques required to use parallel supercomputer versions of our code (James et al., 2004b). In the present paper we discuss some measured properties of the emergent patterns and speculate on their origins and implications.

The features of our animat model relevant to this paper are described in section 2. We present some generated patterns and behaviours in section 4 and some approaches we have taken to make a quantitative analysis of the emergent patterns in section 3. We summarise our analysis of the observed spirals in section 5 and speculate on possible causes and meanings of the spiral phenomena in section 6. We conclude with several further questions to consider and areas for further study in section 7.

2 The Model

Our experimental animat model is a simple predator-prey model; the animats nominally consist of prey (“rabbits”) and predators (“foxes”). Rabbits are considered to have an unlimited amount of food “grass”. In contrast our foxes only predate rabbits; they do not eat grass in the model. A two-phase randomized update method is used to update the system between discrete time-steps (James et al., 2004b). This allows us to establish a well-defined movement phase where only spatial positions are changed and a number-changing phase where animals are born or die.

Our model is based on an open system space. Animats occupy integer coordinates and in

that sense the model is an automata. We have not included an exclusion principle, meaning that more than one agent may inhabit the same space cell at the same time (this is used primarily when agents produce offspring). We find in practice there is very little “agent stacking” or multiple site occupancy. In those rare occasions where there is stacking, typically only two or three agents inhabit the same space and at most ten agents do. We believe this aspect of our model is akin to the self-organised criticality behaviour of Bak’s sandpile model (Bak, 1996).

We have attempted to minimise the number of independent parameters that can be modified in our program to make searching the cross-product of parameters tractable (James et al., 2004a).

The evolutionary rules of our system are relatively simple. They are consulted in the priority order that they appear in the list below. At every time step, each animal executes one rule in order of priority. If rule 1 is executed, all other rules for the animal are ignored. If rule 1 can not be executed, rule 2 is used, and so forth. As previously mentioned, it is possible for animals to be located at the same cellular coordinates as another animal; since our animals have no notion of a third dimension, this just approximates use of a model with finer-grained cells and with longer distances of animal perception. An alternative interpretation is that our animals are Bosonic rather than Fermionic in that more than one animal can simultaneously occupy the same discrete state.

The prey or “rabbit” rules are:

- 1 move away from a fox if the fox is adjacent;
- 2 breed if a rabbit is adjacent and less than 5 rabbits are nearby;
- 3 move towards a rabbit if the rabbit is nearer than 20 spatial units;
- 4 move to a randomly selected adjacent position.

The predator or “fox” rules are as follows:

- 1 eat a rabbit if the rabbit is adjacent;
- 2 move towards a rabbit if it is nearer than 80 spatial units and this fox is hungry;
- 3 breed if a fox is adjacent and less than 3 foxes are nearby;
- 4 move towards a fox if it is nearer than 80 spatial and this fox is not hungry;
- 5 move to a randomly selected adjacent position.

Animals live on an open coordinate system space. There is no “bounding box” or array of cells. Each animal stores its own coordinates. Although we do not enforce conservation of energy in the sense that grass is always available to rabbits, we do ensure transactional semantics to ensure no rabbit is eaten more than once.

We have implemented a pseudo-sexual breeding process which requires less book-keeping than a more realistic paired breeding process. This breeding process consists of each animal that desires to breed producing a random number and checking if this number is less than *half* the current birth rate. If this is the case, a new animal is produced with the same location as the parent (thus stacking the animals). If this is not the case, no new animal is produced but breeding has (unsuccessfully) taken place for purposes of fulfilling the rule and thus no further rules are invoked in the current time step.

It can be seen that the rabbits’ first rule is to attempt to preserve their life by fleeing as soon as a fox moves into an adjacent space. Another important rule essentially embodies a “flocking”

behaviour: when not immediately hungry both predators and prey prefer the company of other animals for the purposes of reproduction. We believe it is the combination of these two agent rules that have most directly contributed to the spiral phenomenon we explore in this paper.

The parameters chosen are somewhat arbitrary. The models seem remarkably insensitive to fine-grained manipulation of the parameters. The model can be initialised as a random inter-mixing of rabbits and foxes or as interleaved blocks of serried rabbits and foxes. The system rapidly loses any resemblance to the starting configuration, and the spiral patterns we observe seem to be a universal attractor of some sort.

3 Methodology

Our original programme of investigation was based on our prejudices that we could formulate a fast animat simulation code that would support a systematic statistical exploration of the parameter space of some simple animat model combinations. Although we are still pursuing this, we have been forced to develop and implement new analysis techniques and software to attempt to characterise these patterns quantitatively.

We coded the model in terms of the parameters in section 2 and batch ran a scan over parameters looking for regimes in which the animal population reached a relatively stable equilibrium point. For the part of parameter space in which the spirals appear we are analysing time-series data to track the development and decay of the spirals. We have developed a prototype configuration editor and statistical measurement program that allows us to visualise the state of the model system and measure the characteristics of the spirals.

The model parameter regime we discuss here has:

- maximum animal lifetime is 50 (foxes) or 20 (rabbits) time steps
- animal perception distance is 80 (foxes) or 20 (rabbits) spatial units
- proximity distance of unity (in same cell)
- crowding limitation of reproduction 3 (foxes) or 5 (rabbits) animals in proximity
- foxes starve if they fail to eat a rabbit in 20 time steps.

These are all **local** parameters that are effectively wired into the animat agents, but can be made locally adaptive. We do impose one **globally calculated** parameter – as a damping system regulation parameter. The global birth rate for each animal is dynamically calculated for the whole population. This is fed into an animal when it makes stochastic breeding choices. This is a useful way to damp our explosive population growth, without which our rabbit animats behave far too much like real rabbits. We believe the effect of computing this globally is to apply a breeding “dampening field”. This rate could be calculated locally by animats, based on their input measurements from their perceived neighbours. We believe a more realistic food consumption model where the “grass” was only replenished at a certain rate would have a similar effect.

Our model does not “conserve energy” in any strict sense and it is open-ended in terms of having an unbounded grass supply for rabbits to consume. For the purposes of the present work this is not dissimilar to studying growth of a bio-sample in a petri dish for the time regime during which there is a plentiful supply of nutrients. We do plan to look further into a process-based model where grass is in limited supply either in time or space. In fact, we are exploring this feature in subsequent work.

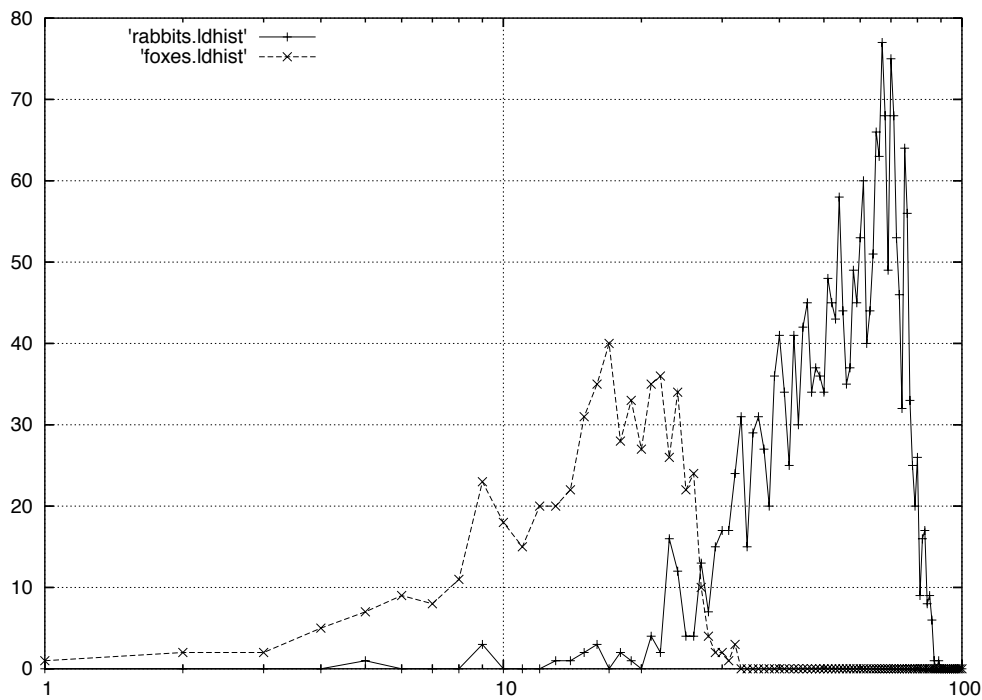


Figure 2. Connectivity graph degree analysis of the configuration shown in figure 1. The distribution functions of animats with a particular degree is shown. Typically rabbits are more highly connected (mean degree of 46, max 89) than foxes (mean degree of 4, max 32). A clustering (proximity) distance of 6 was used for this analysis. The proximity range of 6 is sufficient to regard the illustrated spiral as a single cluster.

We are able to partially automate analysis of particular configurations of our model by identifying clusters of particular animats. Figure 2 shows a distribution of the connectivity degree of the animats in our system. Rabbits tend to live “highly connected” lives with a mean degree of 46. Foxes tend to have a much smaller degree of other foxes in their proximity – in this configuration a typical fox has only around 4 nearby foxes. Of the 2384 animats in this particular configuration: 2277 were rabbits; 557 were foxes. All the rabbits are part of the spiral, whereas only 374 of the foxes are in the spiral. The remaining foxes are in the 3 small clusters to the left, and will die of starvation in the absence of any rabbits.

The spiral patterns appear at various positions and times during the simulation irrespective of detailed start configurations or random number generator sequence. We present some characteristic snapshots of the patterns in the section 4 below. In section 5 we show how the spirals can be measured quantitatively.

4 Selected Results – Some Patterns Generated

We present a number of snapshots of the model in the following figures. The colour versions show rabbits as blue pixels and foxes as red pixels. We record the position of animal corpses as green pixels since these fixed points allow the eye to follow the dynamics of what is otherwise an open system. These colours appear as dark, medium and light grey on a white background in printed versions of this paper.

Figure 3 shows a separation of the animat components from the model configuration of fig-

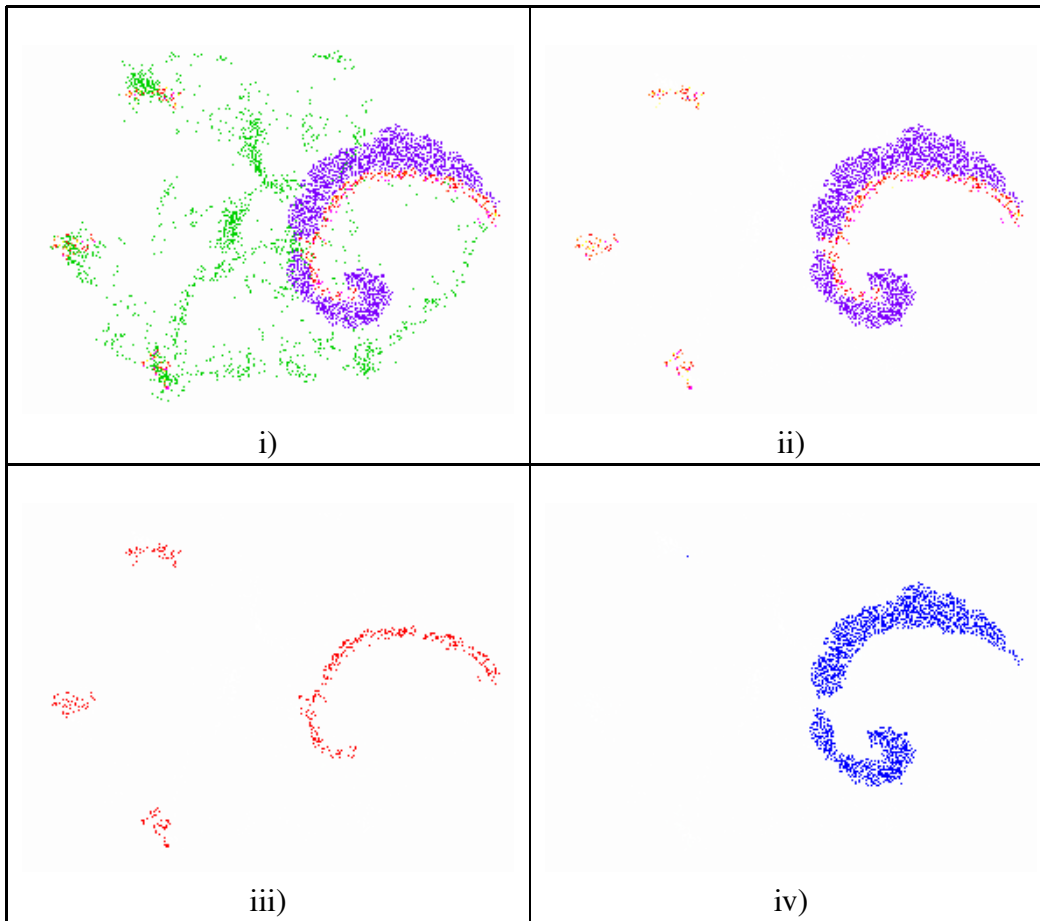


Figure 3. A separation of animat components from the configuration shown in figure 1: i) is the original; ii) the system is shown without the fox corpses; iii) only the live foxes are shown; iv) only the live rabbits are shown.

ure 1. Although animats do form clusters of various shapes, the most common are: circularly symmetric clumps of unmolested rabbits; diffuse clusters of rabbit-less, starving foxes; and sustained “battlefronts” of retreating rabbits and attacking foxes arranged in stationary or rotating spirals with foxes on the inside edge.

Figure 4 shows the essential property of rabbits in the model. They multiply unbounded if not predated and they form circularly-symmetric clumps if left to their own devices. We have chosen a parameter regime of study so that when foxes are added to the system they predate rabbits and the system self-regulates to a large extent. We were surprised that there are parameter regimes where typically rabbit and fox populations **do** rapidly self-regulate and the system does not grow unbounded despite our open coordinate system.

Foxes cannot exist without rabbits. An injected population of isolated foxes will die of starvation without rabbits to feed on. The corpse patterns of foxes as shown in figure 1 form a sparse connected network of battle areas of feeding frenzies where typically foxes over-gorge on rabbits and subsequently die of starvation. The disconnected areas that are clear of fox corpses are where foxes diffused outwards in search of rabbits, and generally died *en masse* along a wavefront at the same starvation age.

Figure 5 shows the populations (global) and (for a particular spiral) of the rabbits and foxes. In general although there is a cyclic pattern of predator-prey population growth and decay in the system population as a whole, it is superposed on a remarkably stable envelope function in

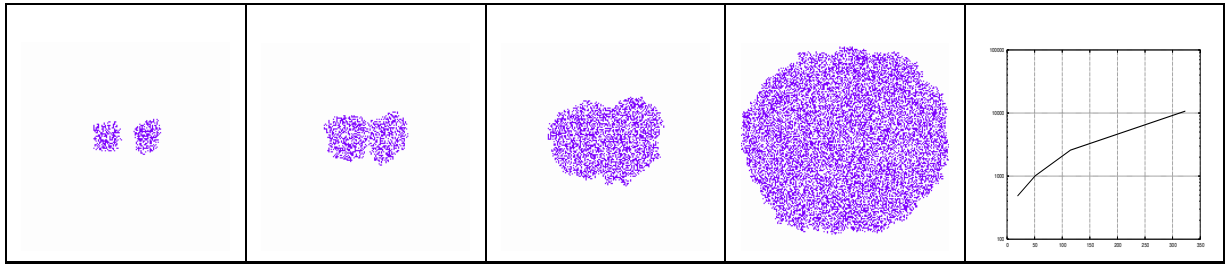


Figure 4. Time lapse sequence showing the consequences of the rabbits' simplistic rules: if there are no nearby foxes, try to be close to other rabbits. This cluster consists solely of rabbits. The graph on the right hand side shows the population of the cluster as time progresses. Population (y-axis) – ranging over three decades (from 484 to over 10,000) – is on a log scale, time (x-axis) – from timesteps zero to 350 – is linear. Rabbits left unmolested will clump in symmetrically growing clusters.

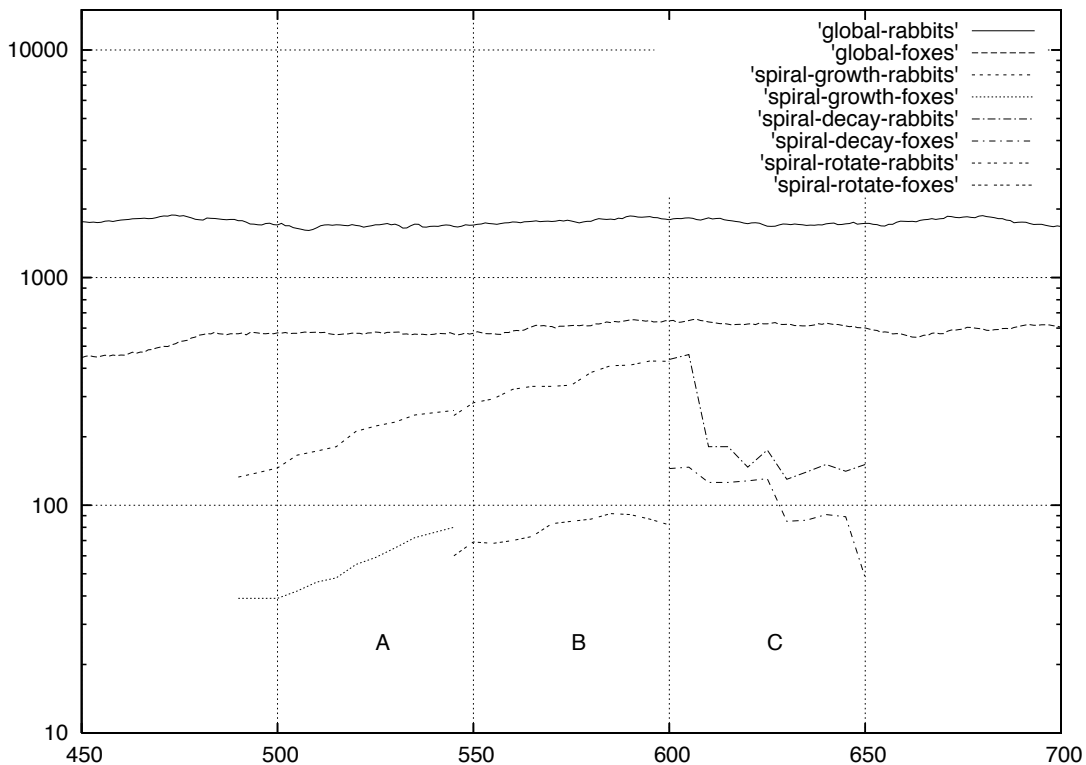


Figure 5. Global population of rabbits and foxes in the system from time steps 450-700, during which time a specimen spiral is in 'A: growth' from steps 500-550, 'B: rotates' from 550-590, and 'C: decays' from 590-650. The upper curves of each pair-set are rabbit populations and the lower are for foxes. Generally growth is characterisable as a bounded exponential-monotonic straight line fit from the log scale plot.

the parameter regime we describe in this paper. Our supercomputer parameter-scan batch runs, originally designed to find regimes of interest, homed in on the parameter set we describe in section 2 as a stable regime. However it was something of a surprise to find the spiral pattern formations that appear to be an attractor for the model in these regimes. The global population measurements give no indication of this phenomenon.

In figures 6, 7 and 8 below, we show snapshots of the regimes of A:growth B:decay and C:rotation respectively.

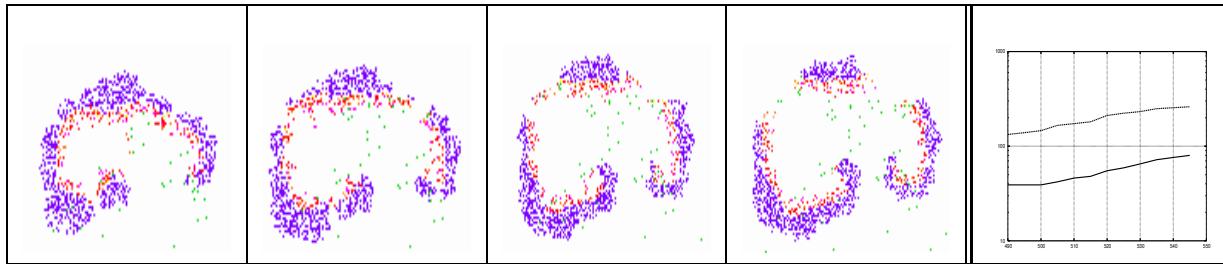


Figure 6. *Time lapse sequence showing the growth of a spiral. Dark pixels (coloured blue) represent rabbits while lighter pixels (coloured red) represent foxes. The lightest pixels (coloured green) represent animal corpses. The four frames on the left show the state of the spiral at steps 500, 520, 535 and 545. The graph on the right shows the trend of the respective rabbit and fox populations in the right-hand spiral for fifty timesteps from step 490. Animal populations – over two decades – are shown in log scale; there are always more rabbits (top curve) in the model than foxes (lower curve).*

Figure 6 shows growth of a spiral pattern in a typical time-lapse sequence. The right hand spiral in the left-most frame is breaking up into two clusters, each of which will continue to spiral independently in different rotational directions. The left hand one is tracked and the populations of rabbits and foxes belonging to it is plotted in the right hand graph. Animal populations from figure 5 are shown on the right hand graph for convenience. Measurements of the radii of of the spiral indicate steady growth constant of approximately 1.1 in 180° .

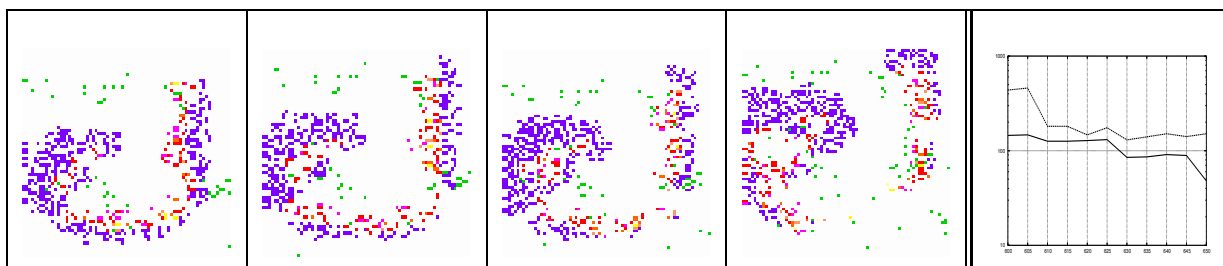


Figure 7. *Time lapse sequence showing the decay of the spiral shown in figure 6 at timesteps 590, 600, 610 and 630. The graph shows the decline in animal population in the right hand spiral as it decays over fifty timesteps. Again, the population is in a log scale over two decades and there are always more rabbits in the spiral than foxes.*

Figure 7 illustrates the eventual break up and decay of the spiral that is shown growing in figure 6. It is harder to characterise a spiral cluster as it breaks up, as it is no longer trivial to identify it. Some manual measurements of typical radii suggest the cluster size scales as a power law.

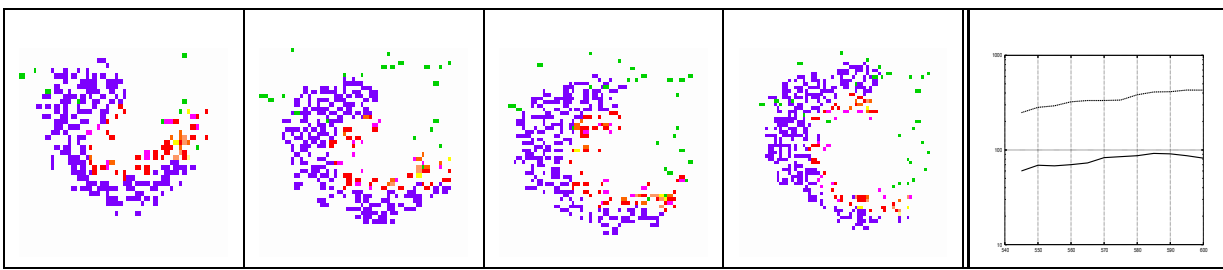


Figure 8. Time lapse sequence showing the rotation of a spiral as the predators’ attacking front herds the rabbits away. The graph shows the respective populations of rabbits and foxes, over two decades in log scale, over fifty (linear) timesteps. Generally, rotating spirals seem to grow more rapidly than stationary ones.

Figure 8 shows the changes in animal populations when a typical spiral is rotating. The rabbits manage to out-breed the foxes and in so doing the cluster slowly rotates. Although the tail is consumed by foxes, the new rabbit growth at the outer rim maintains the size of the spiral and even makes it grow spatially.

timestep	r_1	r_2	ratio
560	9.10	13.0	1.42
575	13.2	19.1	1.45
590	7.10	20.8	2.93

Table 1. Rotational growth of pattern in figure 8. Measured radial distance of the end point (r_1) and at a selected 180° around the spiral (r_2). The ratios allow a decay or growth constant to be fitted.

Table 1 shows some typical radii measured at control points on the spirals. Two points at 180° rotated positions around the spiral are enough to estimate a growth or decay ratio. It is easier to estimate the centre of convergence of a particular spiral and estimate a decay constant from that rather than using the centre of mass in a rotating coordinate frame as the centre.

5 Spiral Analysis

The spiral patterns can be analysed with respect to their characteristic spatial sizes, orientations and animat populations. Our model has no particular rotational symmetries and we observe spirals of both handedness. Small spirals of a few hundred animats form very readily in our model regime and generally will grow exponentially and then decay and break up as other patterns interfere with them. If a spiral is allowed to grow unimpeded without interference from other spirals or clusters, it may end up “chasing its own tail” as it apparently rotates. Clusters commonly spiral inwards and rotate until they are too small spatially to support a recognisable formation. This is not easy to illustrate in snapshots but makes a fascinating movie sequence (Hawick et al., 2004).

We have attempted to characterise the spiral dimensions that typically arise in our model. It is relatively straightforward to fit a centre of mass and estimated radius of gyration for a given spiral.

We define centre of mass $r_M = (x_M, y_M)$ in terms of the mass-averaged coordinates for all

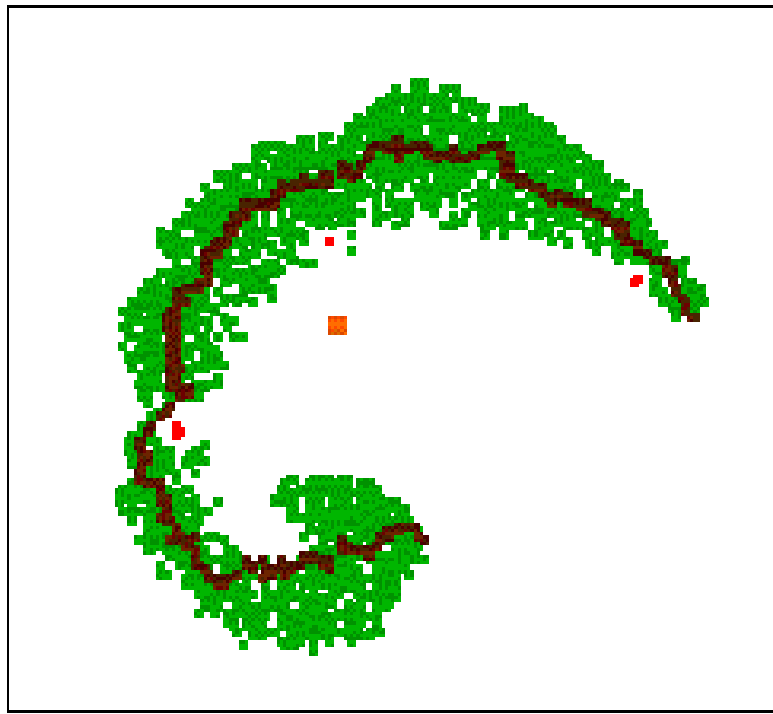


Figure 9. A typical spiral pattern that emerges from a model run, with predators (red pixels) preying on prey (coloured blue). The cluster of prey are attacked from inside by the predators. The drawn line is measured automatically from a graph colouring identification of the particular cluster, location of its centre of mass, and a rotational scan for mean radius.

the animals i a particular cluster j :

$$r_{M_j} = \frac{1}{M_j} \sum_i m_i r_i \quad (1)$$

Measurement of the radius of gyration R_{G_j} for the j 'th cluster gives an indication as to whether it is a spiral or not in comparison with the maximum and minimum radii of the cluster.

Radius of gyration is defined as:

$$R_{G_j} = \frac{1}{M_j} \sqrt{\left(\sum_i m_i r_i^2 \right)} \quad (2)$$

Figure 9 shows a solid line tracking the spiral core line, measured from the animal spatial coordinates. We can identify particular spirals automatically using a graph-colouring algorithm (Gebremedhin and Manne, 2000) and a proximity parameter. Generally we have found that a proximity parameter of 3 is sufficient to identify all the rabbits in a particular spiral. This value is consistent with the rabbit breeding proximity parameter, although we have not yet established a systematic functional dependence.

A convenient parametric equation for a spiral in terms of polar coordinates (R, θ) about a centre point is:

$$R = R_0 e^{-A_0/\theta} \quad (3)$$

Figure 10 shows a means of parameterising spirals in terms of the radius as a function of rotational angle around the centre of mass. A value of $A_0 = \log 2 / \pi$ gives a decay of one half in

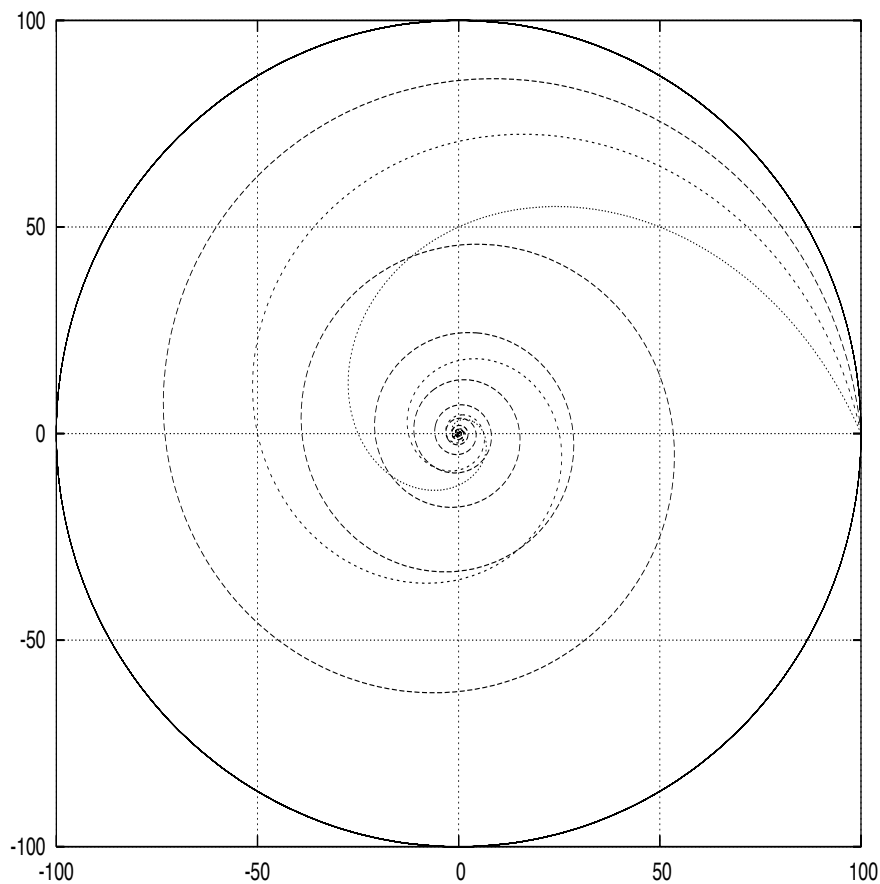


Figure 10. *Spirals characterised by an exponential decay function of the rotational angle. Four curves are shown with different decay constants.*

π radians or 180° . This formula is useful in fitting spiral growth and decay values to estimated radii.

We have not explored model systems capable of supporting spirals of more than two complete rotations, but we do not believe there are fundamental scale limits to these forming in our model.

6 Discussion

We believe the battlefronts formed between rabbits and foxes, and the flanking behaviour as rabbits retreat are similar to those that occur in defensive situations in real-world skirmishes: the attacked troops retreat from the attackers, usually in a direction parallel to the attack vector.

It is interesting to observe the emergent pack-hunting of foxes inside a clump of rabbits. They corrode the rabbit cluster from the inside out. Rabbits will preferentially breed and move on the **outside** of the clump. The curvature direction or convexity of the rabbit-fox front is therefore fixed. Rabbits are always on the “outside”. Rabbits at the edge of a fox-incursion can flank around the attacking foxes by breeding. This gives rise to the beginnings of a spiral. In some cases a double-ended spiral occurs – reminiscent of the attacking “horns of the bull” favoured by some military strategists.

Providing rabbits can reproduce at a sufficient rate, and that foxes do not live too long, the

battlefront troop numbers are sustained and the front simply moves. This, coupled with the flanking behaviour gives rise to the rather beautiful rotating spiral patterns, that we believe are justifiably called emergent patterns from our model.

Our model also illustrates simple herding behaviour. Rabbits need to cluster together for breeding but flee foxes. Individuals rabbits fail to avoid foxes, but the rabbit population as a whole does not collapse (in the right parameter regime). The rabbits retreating behaviour apparently causes the flanking or curvature. This safety in numbers phenomena is familiar from real-world situations.

The language and metaphors that spring to mind to describe our defensive spirals are reminiscent in the rich descriptions of systems in D'Arcy Wentworth Thompson's classic "On Growth and Form" (Thompson, 1942). His chapter on spirals characterises various spiral and whorl patterns in nature including those of various sea shells. Our model has an extra richness in that animals or "mass" can pile up at the same spatial coordinates. Also, our animals are able to **move** in response to the environmental inputs – predation and hunger.

It is not clear why this spiral pattern behaviour emerges as a universal attractor in our model. It seems likely however that it is due to the frontal interplay between the two species. This push-pull behaviour between the two different animal groups' strategies seems important for understanding other real-world behaviour.

At present we have no reason to believe the spiralling behaviour is limited to particular length scales although we have limited our study to date to relatively small manageable simulations of a few thousand animals. Our model is obviously of $O(n^2)$ for n animals, although because of the short-range interactions, there are a number of book-keeping optimisations such as Particle-in-cell (Norton et al., 1995) that could be used to lower the time complexity in practical terms.

7 Conclusions

In conclusion, our predator-prey model has shown some surprising complexity in behaviour, despite the simple rules governing agent behaviour. It has opened up some important questions for further study concerning the effect of open systems, and of multiple spatial occupancy.

It remains to be seen if we can develop an evolutionary framework in which a population of animals with adaptive strategies will "evolve" a strategy gene that corresponds to the particular parameters of our present model. In any case, we believe we have found an interesting emergent behaviour in the form of the defensive spiral pattern generation. While we believe this phenomena worthy of the term, following the discussion in (Ronald et al., 1999b) we hope further study will allow us to discard the epithet "emergent" from our description of these patterns.

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