Well-posedness and Finite Dimensional Approximation for a Modified Camassa-Holm Equation

Robert McLachlan, Xingyou Zhang
(*Institute of Fundamental Sciences, Massey University, New Zealand*)
E-mail: R.McLachlan@massey.ac.nz, x.y.zhang@massey.ac.nz

Abstract

We establish the local well-posedness in $H^s(S)$ with any $s > \frac{7}{2}$ for a modified Camassa-Holm equation derived as the EPDiff equation with respect to the $H^2(S)$ metric, and obtain the global existence of the weak solution in $H^2(S)$ under some sign assumption on the initial values and prove the convergence of the corresponding finite particle approximation method.

Key words Modified Camassa-Holm equation; well-posedness; global existence; particle method.

1 Introduction and Main Results

In this paper, we study the Cauchy problem for a modified Camassa-Holm equation derived as the EPDiff equation with respect to the $H^2(S)$ metric, i.e., the Cauchy problem for

$$(I-\partial_x^2+\partial_x^4)u_t+2\partial_x u\cdot(I-\partial_x^2+\partial_x^4)u+u\cdot(\partial_x-\partial_x^3+\partial_x^5)u=0, \quad x\in S, \tag{1.1}$$

where S is the circle $\mathbb{R}/\mathbb{N}$.

Many works are devoted to the study of the so-called Camassa-Holm equation

$$u_t-\partial_x^2\partial_t u+3uu_x=2u_x\partial_x^2 u+u\partial_x^3 u \tag{1.2}$$

since it was derived by R. Camassa and D. Holm [3] and B. Fuchssteiner and A. Fokas [8] as a bi-Hamiltonian extension of KdV equation: to mention a few, Arnold and Khesin [1], Constantin and McKean [7], McKean [17], and the references therein. Local well-posedness for (1.2) was discussed by Constantin [5] , Constantin and Escher [6] for the initial data in $H^s(S)$ with $s \geq 4$ and $s \geq 3$ respectively, and by Misiołek [16] with $s > 3/2$. Local well-posedness in the non-periodic case was proved for the intial data in $H^s(\mathbb{R})$ with $s > 3/2$ by Li and Olver [15] and Rodriguez-Blanco [18]. It is worthwhile to mention that Xin and Zhang [19] proved the global existence of the *weak solution* in the *energy space* $H^1(\mathbb{R})$ without any sign conditions on the initial value, and the uniqueness of this *weak solution* is obtained under some restrictive conditions on the solution [20].

One point of view is to regard the Camassa-Holm equation as the equation of the geodesic flow associated to $H^1(S)$ metric on the diffeo-group Diff(S) of the circle [13] [14]. If one replaces $H^1(S)$ metric with $H^2(S)$ metric, then the corresponding EPDiff equation is (1.1). The Camassa-Holm equation, however, is completely integrable, hence it is possible that some of the existence results are related to its integrability. But the EPDiff equation in Diff(S) for H^k metric, $k > 1$, is not known or believed to be integrable, so we may expect some different dynamics. We here consider the case $k = 2$ as a first step towards a more general study of geodesic in Diff(S).

Our main results are

Theorem 1.1 *Let $u_0 \in H^s(S)$, $s > 7/2$. Then, there exist $T > 0$ depending on $||u_0||_s$, and unique solution u satisfying (1.1) in the distribution sense such that*

$$u \in C([0, T], H^s) \cap C^1([0, T], H^{1/2}).$$

Moreover, the map $u_0 \in H^s \to u \in C([0, T], H^s)$ is continuous.

Theorem 1.2 *Suppose $u_0 \in H^s(S)$, $s > 7/2$, and that $m_0 \equiv (I - \partial_x^2 + \partial_x^4)u_0 \geq 0$ (or $(I - \partial_x^2 + \partial_x^4)u_0 \leq 0$). Then Equation (1.1) is globally well posed in $H^s(S)$.*

Theorem 1.3 *$u_0 \in H^2(S)$, $m_0 = (I - \partial_x^2 + \partial_x^4)u_0$ is a positive Radon measure on S. Then there exists a unique global* weak *solution $u \in C([0, \infty); H^2)$ of (1.1). And for this solution, if letting $m = (I - \partial_x^2 + \partial_x^4)u$, we have*

$$\int_S u \mathrm{d}x = \int_S u_0 \mathrm{d}x, \qquad \int_S m^{1/2} \mathrm{d}x = \int_S m_0^{1/2} \mathrm{d}x,$$

$$\int_S (u^2 + u_x^2 + u_{xx}^2) \mathrm{d}x = \int_S (u_0^2 + u_{0x}^2 + u_{0xx}^2) \mathrm{d}x.$$

This paper is organized as follows: § 2 is a quick review of Kato's theory, and § 3 discusses the local well-posedness, and § 4 is devoted to the global existence under some sign conditions on the initial momentum, while in § 5, we discusses the global existence of the weak solution. The so-called *particle method* is very important and efficient in the numerical simulation of the Camassa-Holm equation, so, in § 6, after rewriting (1.1) as (6.16) via an appropriate transformation (6.8)(6.12) of unknown variables, we approximate (6.16) by a system of ODEs (6.17) and prove the convergence of the solution of (6.17) to that of (6.16).

Let us finish this section by introducing some notations: $||\cdot||_X$ for the norm in the Banach space X, $\mathcal{B}(X,Y)$ denotes the space of all bounded linear operators from X to Y($\mathcal{B}(X)$ if $X=Y$; $\partial=\partial_x=\frac{\partial}{\partial x}$; $\Lambda_2^s=(I-\partial_x^2)^{s/2}, \Lambda_4^s=(I-\partial_x^2+\partial_x^4)^{s/4}$, $s\in\mathbb{R}$; $H^s=H^s(S)$ with norm $||\cdot||_{H^s}=||\cdot||_s$ and $\langle\ ,\ \rangle_s$ for its inner product; $H^\infty=\bigcap_{s\geq 0}H^s$; $[A,B]$ denotes the commutator of the linear operators A and B.

2 Review of Kato's Theory

Our local well-posedness is based on Kato's theory [11], so we review it here in this section.

Consider the abstract Cauchy problem

$$(C)\quad \begin{cases} \frac{\partial u}{\partial t}+A(u)u=f(u)\in\ X, & t\geq 0,\\ u(0)=u_0\in Y, \end{cases} \tag{2.1}$$

where $A(u)$ is a linear operator depending on the unknown u. Some assumptions on (C):

(X) X and Y are Hilbert spaces where $Y\subset X$ dense and the inclusion continuous, and there is an isomorphism S from Y to X such that $||\omega||_Y=||S\omega||_X$ for all $\omega\in Y$.

(A_1) Let $W\subset Y$ be an open ball centered at 0. The linear operator $A(u)$ belongs to $G(X,1,\beta)$ where β is a real number, ie, $A(u)$ is quasi-m-accretive:

1. $\langle A\omega,\omega\rangle_X\geq-\beta||\omega||_X^2$, $\forall\ \omega\in\mathcal{D}(A)$, the domain of A.
2. $(A+\lambda)$ is onto for some (all) $\lambda>\beta$.

(A_2) The map

$$\omega\in W\to B(\omega)=[S,A(\omega)]S^{-1}\in\mathcal{B}(X) \tag{2.2}$$

is uniformly bounded and Lipschitz continuous, i.e., there exist constants $\lambda_1, \mu_1 > 0$ such that

$$||B(\omega)||_{\mathcal{B}(X)} \leq \lambda_1, \quad ||B(\omega) - B(\nu)||_{\mathcal{B}(X)} \leq \mu_1 ||\omega - \nu||_Y$$

for all $\omega, \nu \in W$.

(A_3) $Y \subseteq \bigcap_{\omega \in W} \mathcal{D}(A(\omega))$, so that $A(\omega)|_Y \in \mathcal{B}(Y, X)$ by the Closed Graph Theorem. Moreover, there exists $\mu_2 > 0$ such that, for all $\omega, \nu \in W$, we have

$$||A(\omega) - A(\nu)||_{\mathcal{B}(Y,X)} \leq \mu_2 ||\omega - \nu||_X.$$

(f_1) $f : W \to Y$ is bounded and there exist a constant $\mu_3 > 0$ such that

$$||f(\omega) - f(\nu)||_X \leq \mu_3 ||\omega - \nu||_X, \quad \forall\, \omega, \nu \in W,$$

$$||f(\omega) - f(\nu)||_Y \leq \mu_3 ||\omega - \nu||_Y, \quad \forall\, \omega, \nu \in W,$$

Theorem 2.1 (Kato [11]) *Under assumptions above on (C), there exists a $T > 0$ such that there exits a unique solution $u \in C([0,T], Y) \cap C^1([0,T], X)$ to (C). Moreover, the map $u_0 \in Y \to u \in C([0,T], Y)$ is continuous in the following sense: suppose*

$$\lim_{n\to\infty} ||A_n(\omega) - A_\infty(\omega)||_{\mathcal{B}(Y,X)} = 0, \quad \lim_{n\to\infty} ||B_n(\omega) - B_\infty(\omega)||_{\mathcal{B}(X)} = 0,$$

$$\lim_{n\to\infty} ||f_n(\omega) - f_\infty(\omega)||_Y = 0, \quad \lim_{n\to\infty} ||u_{0,n}(\omega) - u_{0,\infty}(\omega)||_Y = 0,$$

and consider the Cauchy problems

$$(C_n) \quad \begin{cases} \frac{\partial u_n}{\partial t} + A_n(u_n)u_n = f_n(u_n) \in X, & t \geq 0, \\ u_n(0) = u_{0,n} \in Y, & n \in \mathbb{Z} \cup \{\infty\}. \end{cases} \tag{2.3}$$

Suppose the assumptions above hold also for (C_n) with the same X, Y, S, W and the constants β, λ_i, μ_i are independent of n. Let T_n be the time of existence of u_n. Then all u_n, with n large enough, can be extended to $[0, T_\infty]$ and

$$\lim_{n\to\infty} ||u_n(t) - u_\infty(t)||_{C([0,T_\infty];Y)} = 0.$$

The proof of Theorem 2.1 can be found in [11].

Lemma 2.1 [11] *Let s, t be real numbers such that $-s < t \leq s$, then*

$$C||f||_s||g||_t \geq \begin{cases} ||fg||_t & \text{if} \quad s > m/2, \\ ||fg||_{s+t-m/2} & \text{if} \quad s < m/2, \end{cases} \tag{2.4}$$

where C is a positive constant depending on s, t, m.

The similar proof of Lemmas A.2 and A.3 (with modifications) in [11] will give

Lemma 2.2 *If $s > 1/2 + 1$, then*

$$||[\Lambda_4^s, f]\Lambda_4^{1-s}|| \leq C||f'||_{s-1}, \tag{2.5}$$

where $|| \cdot ||$ on the left denotes the operator norm in $L^2(S)$.

Lemma 2.3 *Let $m \geq 3$. Then*

$$||[\Lambda_4^s, f]\Lambda_4^{-s}|| \leq \begin{cases} C||f'||_{m/2-1} & \text{if} \quad 0 \leq s < m/2, \\ C||f||_{s-1} & \text{if} \quad s > m/2. \end{cases} \tag{2.6}$$

3 Local Well-posedness

We can rewrite (1.1) in two ways:

$$\begin{cases} m_t = -um_x - 2mu_x, & x \in S, \ t \in \mathbb{R}, \\ m(x, 0) = m_0(x) = \Lambda_4^4 u_0(x), \end{cases} \tag{3.1}$$

where $m = \Lambda_4^4 u = (I - \partial_x^2 + \partial_x^4)u, \ \Lambda_4^s = (I - \partial_x^2 + \partial_x^4)^{\frac{s}{4}}$. Or

$$\begin{cases} u_t = -uu_x - \partial_x \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u), & x \in S, \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \tag{3.2}$$

In order to prove the local well-posedness Theorem 1.1, we need some preliminary lemmas.

Lemma 3.1 [18] *The operator $A(u) = u\partial_x$, with $u \in H^s, \ s > \frac{3}{2}$ belongs to $G(H^{1/2}, 1, \beta)$.*

A similar proof as in [18] with some modifications gives

Lemma 3.2 $B(u) = [\Lambda_4^{s-1/2}, u\partial_x]\Lambda_4^{1/2-s} \in \mathcal{B}(H^{1/2})$ *for $u \in H^s, \quad s > 3/2$.*

Lemma 3.3 [18]

(a) $H^s \subset \mathcal{D}(u\partial_x) = \{f \in H^{1/2} : u\partial_x f \in H^{1/2}\},\ s > 3/2.$

(b) $u\partial_x \in \mathcal{B}(H^s, H^{1/2}),\ s > 3/2.$

(c) $||u\partial_x - v\partial_x||_{\mathcal{B}(H^s,H^{1/2})} \leq C||u - v||_{1/2}.$

Lemma 3.4 *Let* $f(u) = -\partial_x\Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u),\ s > 7/2,$ *then*

(a) $||f(u) - f(v)||_{1/2} \leq C||u - v||_{1/2}.$

(b) $||f(u) - f(v)||_s \leq C||u - v||_s.$

Proof **(a)** We need only to verify that

$$||\partial_x\Lambda_4^{-4}(\frac{7}{2}u_{xx}^2 + 3u_x\partial_x^3 u - \frac{7}{2}v_{xx}^2 - 3v_x\partial_x^3 u)||_{1/2} \leq C||u - v||_{1/2},$$

for the corresponding inequality for the other two terms is easy to verify.

$$\begin{aligned} & ||\partial_x\Lambda_4^{-4}(u_{xx}^2 - v_{xx}^2)||_{1/2} \leq C||\partial_x^2(u+v)\partial_x^2(u-v)||_{-5/2} \\ \leq\ & C||\partial_x^2(u+v)||_{L^\infty}||\partial_x^2(u-v)||_{-5/2} \leq C||u+v||_{5/2}||u-v||_{-1/2} \\ \leq\ & C||u-v||_{1/2}. \end{aligned}$$

$$\begin{aligned} & ||\partial_x\Lambda_4^{-4}(u_x\partial_x^3 u - v_x\partial_x^3 v)||_{1/2} \\ \leq\ & C||u_x\partial_x^3 u - v_x\partial_x^3 v||_{-5/2} = C||u_x\partial_x^3 u - u_x\partial_x^3 v + u_x\partial_x^3 v - v_x\partial_x^3 v||_{-5/2} \\ \leq\ & C||u_x||_{L^\infty}||u-v||_{1/2} + C||\partial_x^3 v||_{L^\infty}||u_x - v_x||_{-5/2} \\ \leq\ & C||u-v||_{1/2}. \end{aligned}$$

(b) Similar situation as in **(a)**.

$$\begin{aligned} & ||\partial_x\Lambda_4^{-4}(u_{xx}^2 - v_{xx}^2)||_s \leq C||\partial_x^2(u+v)\partial_x^2(u-v)||_{s-3} \\ \leq\ & C||\partial_x^2(u+v)||_{s-3}||\partial_x^2(u-v)||_{s-3} \leq C||u+v||_{s-1}||u-v||_{s-1} \\ \leq\ & C||u-v||_s, \end{aligned}$$

$$\begin{aligned} & ||\partial_x\Lambda_4^{-4}(u_x\partial_x^3 u - v_x\partial_x^3 v)||_s \\ \leq\ & C||u_x\partial_x^3 u - v_x\partial_x^3 v||_{s-3} = C||u_x\partial_x^3 u - u_x\partial_x^3 v + u_x\partial_x^3 v - v_x\partial_x^3 v||_s \\ \leq\ & C||u||_{s-2}||u-v||_s + C||v||_s||u_x - v_x||_{s-2} \\ \leq\ & C||u-v||_s, \end{aligned}$$

here we have used the fact that H^s is a Banach algebra for $s > 1/2$.

Proof of Theorem 1.1 Now Theorem 1.1 is just a direct consequence of Theorem 2.1 and the above Lemmas.

4 Global Existence

In this section, we will prove Theorems 1.2 and 1.3.

Lemma 4.1 *Let* $u(x,t)$ *be the solution to (1.1) with* $u_0 \in H^\infty$, *and suppose that* $m_0 = (1-\partial_x^2+\partial_x^4)u_0 \geq 0$ (*or* ≤ 0). *Then* $m = (1-\partial_x^2+\partial_x^4)u \geq 0$ (*or respectively* ≤ 0). *Moreover, if* $m \geq 0$, *then*

$$\int_S m^{1/2}\mathrm{d}x = \int_S m_0^{1/2}\mathrm{d}x.$$

Proof The similar argument as that of Lemma 3.3 in [6] gives the proof and we omit it here.

Lemma 4.2 *Let* $u_0 \in H^s(S),\ s > 7/2$ *and* $m_0 = (1-\partial_x^2+\partial_x^4)u_0 \geq 0$(*or* ≤ 0), *then* $\exists K > 0$ *such that* $||u_{xxx}||_{L^\infty} \leq K$.

Proof At first, we assume that $u_0 \in H^\infty$, u solves (1.1), then it is easy to know that $||u||_{L^2}^2 + ||u_x||_{L^2}^2 + ||u_{xx}||_{L^2}^2$ conserves as long as u exists as a solution to (1.1). From Lemma 4.1, we have $m = \Lambda_4^4 u \geq 0$(or ≤ 0). Let $x_0 \in S$ satisfy $u_{xxx}(x_0) = 0$, then $\forall y \in S$, we have

$$\begin{aligned} u_{xxx}(y) &= \int_{x_0}^{y} \partial_x^4 u \mathrm{d}x = \int_{x_0}^{y} (u - \partial_x^2 u + \partial_x^4 u)\mathrm{d}x - \int_{x_0}^{y} (u - \partial_x^2 u)\mathrm{d}x \\ &\leq \int_S m\mathrm{d}x + ||u||_{L^1} + ||u_{xx}||_{L^1} = \int_S m_0 \mathrm{d}x + ||u||_{L^1} + ||u_{xx}||_{L^1} \\ &\leq \int_S m_0 \mathrm{d}x + ||u||_{L^2} + ||u_{xx}||_{L^2} \leq K, \end{aligned} \tag{4.1}$$

where K depends on m_0 and $||u_0||_{H^2}$. Similarly, we have

$$-u_{xxx}(y) = \int_y^{x_0+1} \partial_x^4 u \mathrm{d}x \leq K.$$

so far we have proved the Lemma for $u_0 \in H^\infty$. A standard approximation can give the proof for $u_0 \in H^s(S),\ s > 7/2$.

Lemma 4.3 *Assume the conditions in Theorem 1.2 hold, then* $||u(t)||_{H^s}$ *is finite for any* $0 < t < \infty$.

Proof Apply Λ_4^s to $u_t = -uu_x - f(u)$, where $f(u) = \partial_x \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u)$, and multiply by $\Lambda_4^s u$ and then integrate over S, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}||u||_s^2 = -2\langle u, uu_x\rangle_s + \langle u, f(u)\rangle_s \tag{4.2}$$

By the Kato-Ponce inequality [12], we have

$$|\langle u, uu_x\rangle_s| \leq C_s ||u_x||_{L^\infty} ||u||_s^2. \tag{4.3}$$

The Cauchy inequality gives

$$|\langle u, f(u)\rangle_s| \leq ||u||_s ||f(u)||_s, \tag{4.4}$$

and

$$\begin{aligned} ||f(u)||_s &\leq C||u^2 + \tfrac{1}{2}u_x^2 - \tfrac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u||_{H^{s-3}} \\ &\leq C(||u^2||_{s-3} + ||u_x^2||_{s-3} + ||u_{xx}^2||_{s-3} + ||u_x\partial_x^3 u||_{s-3}) \\ &\leq C(||u||_{L^\infty}||u||_{s-3} + ||u_x||_{L^\infty}||u_x||_{s-3} \\ &\qquad + ||u_{xx}||_{L^\infty}||u_{xx}||_{s-3} + ||u_x||_{L^\infty}||\partial_x^3 u||_{s-3} + ||\partial_x^3 u||_{L^\infty}||u_x||_{s-3}) \\ &\leq C||u||_s, \end{aligned} \tag{4.5}$$

where we used again the Kato-Ponce inequality [12] and Lemma 4.2. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||u||_s^2 \leq C||u||_s^2, \tag{4.6}$$

and so the Gronwall's inequality completes the proof of Lemma.

Proof of Theorem 1.2 Theorem 1.2 is a direct consequence of Lemma 4.3 above.

5 Global Existence of Weak Solution

In this section we will prove Theorem 1.4. Equation (3.2) can be rewritten as

$$\begin{cases} u_t + F(u)_x = 0, \quad x \in S, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{5.1}$$

where $F(u) = \frac{1}{2}u^2 + \Lambda_4^{-4}(u^2 + \frac{1}{2}u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x\partial_x^3 u)$.

Definition 5.1 Let $u_0 \in H^2(S)$. A function $u : [0, +\infty) \times S \to \mathbb{R}$ is called a *global weak solution* to (5.1) if $u \in C([0, \infty); H^2)$ and $\forall\ T > 0$, we have

$$\int_0^T\!\!\int_S (u\varphi_t + F(u)\varphi_x)\mathrm{d}x\mathrm{d}t + \int_S u_0(x)\varphi(0,x)\mathrm{d}x = 0, \quad \forall\ \varphi \in C^{1,c}([0,T) \times S), \tag{5.2}$$

where $C^{1,c}([0,T) \times S)$ is the set of all first order smooth function with compact support in $[0,T) \times S$.

Proof of Theorem 1.3 Let $\theta \equiv |y_0|_{\mathcal{M}} = |u_0 - \partial_x^2 u_0 + \partial_x^4 u_0|_{\mathcal{M}}$, then by Lemma 5.2 in [6], there exist $y_0^n \in C^\infty(S),\ y_0^n \geq 0$ such that $|y_0^n|_{L^1} \leq \theta$ and $y_0^n \to y_0\ \mathcal{D}'(S)$. Denote $u_0^n = \Lambda_4^{-4} y_0^n$, then $y_0^n = u_0^n - \partial_x^2 u_0^n + \partial_x^4 u_0^n$ and

$$\begin{aligned} ||u_0^n||_{H^2}^2 &= \int_S |u_0^n|^2 + |u_{0x}^n|^2 + |u_{0xx}^n|^2 \mathrm{d}x = |\int_S y_0^n \cdot u_0^n \mathrm{d}x| \\ &\leq ||y_0^n||_{L^1} ||u_0^n||_{L^\infty} \leq C||y_0^n||_{L^1} ||u_0^n||_{H^1}, \end{aligned} \tag{5.3}$$

which implies that

$$||u_0^n||_{H^2}^2 = \int_S |u_0^n|^2 + |u_{0x}^n|^2 + |u_{0xx}^n|^2 \mathrm{d}x \leq C||y_0^n||_{L^1}^2 \leq C\theta^2. \tag{5.4}$$

Then by the Theorems 1.1 and 1.2 for the smooth initial value $u_0^n(x)$ there exists a unique solution to (5.1) $u^n \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$. And

$$||u^n(t)||_{H^2} = ||u_0^n||_{H^2} \leq C \ \text{ and } \ |y^n(t)|_{L^1} = |y_0^n(t)|_{L^1} \leq C,$$

if we denote $y^n = u^n - \partial_x^2 u^n + \partial_x^4 u^n$, where C is a constant independent of n. So

$$||\partial_x^4 u^n||_{L^1} \leq ||u^n||_{L^1} + ||\partial_x^2 u^n||_{L^1} + ||y^n(t)||_{L^1} \leq C$$

and

$$||\partial_x^3 u^n||_{L^\infty} \leq C, \qquad \text{with } C \text{ independent of } n.$$

So $\{u^n(t)\}$ is a compact set in $H^2(S)$ for any $t \geq 0$, and $\forall\ n \geq 1$. On the other hand, $||\frac{\mathrm{d}u^n}{\mathrm{d}t}||_{H^2} = ||F(u^n)_x||_{H^2}$ can be estimated as follows:

$$\begin{aligned} ||[(u^n)^2]_x||_{H^2} &= 2||u^n u_x^n||_{H^2} \leq C(||u^n u_x^n||_{L^2} + ||u_{xx}^n u_x^n||_{L^2} + ||u^n u_{xxx}^n||_{L^2}) \\ &\leq C||\partial_x^3 u^n||_{L^2} \leq C||\partial_x^3 u^n||_{L^\infty} \leq C, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &||\partial_x \Lambda_4^{-4}(v^2 + \tfrac{1}{2}v_x^2 - \tfrac{7}{2}v_{xx}^2 - 3v_x\partial_x^3 v)||_{H^2} \\ &\leq C||v^2 + \tfrac{1}{2}v_x^2 - \tfrac{7}{2}v_{xx}^2 - 3v_x\partial_x^3 v||_{H^{-1}} \leq C||v||_{H^2} \\ &\leq C \quad \text{if} \ \ v = u^n. \end{aligned} \tag{5.6}$$

So $||\frac{\mathrm{d}u^n}{\mathrm{d}t}||_{H^2} = ||F(u^n)_x||_{H^2} \leq C$ with C independent of t and n. Therefore $\{u^n(t)\}_{n\geq 1} \subset C([0,\infty);H^2)$ is a compact subset. And $\exists\ u \in C([0,\infty);H^2)$ and $n_k \to \infty$ such that

$$u^{n_k} \to u \quad \text{in } C([0,\infty);H^2),$$

with

$$||u(t)-u(s)||_{H^2} \leq C|t-s|, \quad \forall\ t,s \geq 0,$$

$$u(0) = u_0.$$

Taking $n_k \to \infty$ in

$$\int_0^T\!\!\int_S (u^{n_k}\varphi_t + F(u^{n_k})\varphi_x)\mathrm{d}x\mathrm{d}t + \int_S u_0^{n_k}(x)\varphi(0,x)\mathrm{d}x = 0, \quad \forall\ \varphi \in C^{1,c}([0,T)\times S) \tag{5.7}$$

yields that $u \in C([0,\infty);H^2)$ is the weak solution to (5.1). A similar argument as in [6] can give the uniqueness of this weak solution.

6 Particle method and finite dimensional systems

In the numerical simulation of Camassa-Holm equation, the so-called particle method is important and efficient [4]. In this section, we will show that it applies to the modified CH equation too.

Let $G(x)$ be the Green's function for the operator $\Lambda_4^4 = I - \partial_x^2 + \partial_x^4$ acting on $H^\infty(S)$, then from

$$(I - \partial_x^2 + \partial_x^4)G(x) = \delta(x) = \sum_{n=-\infty}^{\infty} e^{2\pi nix}, \tag{6.1}$$

we have

$$G(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+(2\pi n)^2+(2\pi n)^4}e^{2\pi inx} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{1+(2\pi n)^2+(2\pi n)^4}\cos(2\pi nx) \quad x \in S. \tag{6.2}$$

Obviously, for any $0 \leq \varepsilon < 1,\ G(x) \in C^{2+\varepsilon}(S)$.

Now that

$$u(x,t) = \int_0^1 G(x-y)m(y,t)\mathrm{d}y \tag{6.3}$$

and $m_0 \geq c > 0$, so $m(x,t) \geq 0$ for any $t > 0$, so (3.1) can be rewritten as

$$(m^{1/2})_t = -(um^{1/2})_x\ . \tag{6.4}$$

Let

$$w(x,t)=\int_0^x m(y,t)^{1/2}\mathrm{d}y, \tag{6.5}$$

then $w_{xt}+(uw_x)_x=0,\ \forall\ x\in S$. So there exists a function $g(t)$ such that

$$w_t+uw_x=g(t) \qquad \forall\ x\in S. \tag{6.6}$$

Introducing characteristic curves

$$x=q(\xi,t), \qquad q(\xi,0)=\xi, \tag{6.7}$$

then Equation (6.6) reads as

$$\dot{x}=\dot{q}=u(q,t), \qquad \dot{w}=g, \tag{6.8}$$

where $\dot{f}$ denotes the total derivative

$$\dot{f}\equiv\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)f.$$

From (6.8), we have

$$w(q(\xi,t),t)=\int_0^t g(s)\mathrm{d}s+w(\xi,0), \qquad \text{and so} \qquad \frac{\mathrm{d}w}{\mathrm{d}\xi}=\frac{\mathrm{d}w_0}{\mathrm{d}\xi}, \tag{6.9}$$

where $w_0(\xi)\equiv w(\xi,0)$. Combining (6.3) with the first equation of (6.8) gives

$$u(q(\xi,t),t)=\dot{q}(\xi,t)=\int_0^1 G(q(\xi,t)-q(\eta,t))m(q(\eta,t),t)\frac{\partial q(\eta,t)}{\partial\eta}\mathrm{d}\eta. \tag{6.10}$$

From (6.5) and (6.9) we have

$$m(q(\xi,t),t)=\left(\frac{\frac{\mathrm{d}w}{\mathrm{d}\xi}}{\frac{\partial q(\xi,t)}{\partial\xi}}\right)^2=\left(\frac{\frac{\mathrm{d}w_0}{\mathrm{d}\xi}}{\frac{\partial q(\xi,t)}{\partial\xi}}\right)^2. \tag{6.11}$$

Introducing an auxiliary function

$$p(\xi,t)=m(q(\xi,t),t)\frac{\partial q(\xi,t)}{\partial\xi}=\frac{(w_0'(\xi))^2}{\frac{\partial q(\xi,t)}{\partial\xi}}, \tag{6.12}$$

then we have

$$\dot{p}(\xi,t)=-p(\xi,t)\int_0^1 G'(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta, \tag{6.13}$$

and (6.10) becomes

$$\dot{q}(\xi,t)=\int_0^1 G(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta. \tag{6.14}$$

The solution to (6.13)(6.14) with the initial conditions $q(\xi,0)=\xi,\ p(\xi,0)=(w_0'(\xi))^2$ determines the characteristic curves $x=q(\xi,t)$. On the other hand, (6.13)(6.14) is a Hamiltonian system with

$$H=\frac{1}{2}\int_{S\times S}G(q(\xi,t)-q(\eta,t))p(\xi,t)p(\eta,t)\mathrm{d}\xi\mathrm{d}\eta.$$

Integrating directly (6.13) yields that

$$P\equiv\int_0^1 p(\xi,t)\mathrm{d}\xi$$

is independent of time t because $G'(x)$ is symmetric with respect to $x=1/2$.

From (6.13) we have $|\dot{p}(\xi,t)/p(\xi,t)|\le c_1P$, where $c_1=|G'(x)|_{L^\infty}$. So

$$p(\xi,0)e^{-c_1Pt}\le p(\xi,t)\le p(\xi,0)e^{c_1Pt},\qquad \forall\xi\in S \tag{6.15}$$

In order to approximate the Hamiltonian equations (6.13)(6.14):

$$\left\{\begin{array}{rcl}\dot{q}(\xi,t)&=&\displaystyle\int_0^1 G(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta,\\ \dot{p}(\xi,t)&=&\displaystyle -p(\xi,t)\int_0^1 G'(q(\xi,t)-q(\eta,t))p(\eta,t)\mathrm{d}\eta,\end{array}\right. \tag{6.16}$$

we can use the so-called *particle method*, which takes

$$q_i(t)\equiv q_i(\xi_i,t),\qquad p_i(t)\equiv p_i(\xi_i,t),\quad i\in\mathbb{N}$$

as position coordinates and momenta, and if, for example, q and p are evaluated at points $\xi_i=ih,\ i=1,2,\cdots,N$, obtain the (finite dimensional) discretised version of (6.16):

$$\left\{\begin{array}{rcll}\dot{q}_i&=&\displaystyle h\sum_{j=1}^N G(q_i-q_j)p_j,&\qquad i=1,2,\cdots,N,\\ \dot{p}_i&=&\displaystyle -hp_i\sum_{j=1}^N G'(q_i-q_j)p_j,&\qquad i=1,2,\cdots,N.\end{array}\right. \tag{6.17}$$

Compared with other classical numerical methods for PDE, the main advantage of the particle method is that it preserves the Hamiltonian structure of (6.16) and so we can use the geometric integration [9] to simulate it numerically.

Proposition 6.1 *For any* $l_p>0$, *the right hand side of (6.17) is Lipschitz continuous on* $(q,p)\in D$, *where* $D\subset\mathbb{R}^{2N}$ *is the set of points* $(q,p)=(q_1,q_2,\cdots,q_N;p_1,p_2,\cdots,p_N)$:

$$0\le q_i\le 1,\quad i=1,2,\cdots,N,\quad \max_i|p_i|<l_p<\infty.$$

So the system of ODEs (6.17) admits a unique local solution.

Proof We demonstrate only for the first equation of (6.17) and omit the second one for it is analogous. Let $(p, q),\ (\tilde{p}, \tilde{q}) \in D \subset \mathbb{R}^{2N},\ c_0 = \max_{x\in S} |G(x)|,\ c_1 = \max_{x\in S} |G'(x)|$, then

$$\begin{aligned}
&\left| h\sum_{j=1}^{N} G(q_i - q_j)p_j - h\sum_{j=1}^{N} G(\tilde{q}_i - \tilde{q}_j)\tilde{p}_j \right| \\
\le\ & \left| h\sum_{j=1}^{N} G(q_i - q_j)p_j - h\sum_{j=1}^{N} G(q_i - q_j)\tilde{p}_j + h\sum_{j=1}^{N} G(q_i - q_j)\tilde{p}_j - h\sum_{j=1}^{N} G(\tilde{q}_i - \tilde{q}_j)\tilde{p}_j \right| \\
\le\ & \left| h\sum_{j=1}^{N} G(q_i - q_j)p_j - h\sum_{j=1}^{N} G(q_i - q_j)\tilde{p}_j \right| + \left| h\sum_{j=1}^{N} G(q_i - q_j)\tilde{p}_j - h\sum_{j=1}^{N} G(\tilde{q}_i - \tilde{q}_j)\tilde{p}_j \right| \\
\le\ & c_0 h\sum_{j=1}^{N} |p_j - \tilde{p}_j| + h\sum_{j=1}^{N} |G(q_i - q_j) - G(\tilde{q}_i - \tilde{q}_j)|\,\tilde{p}_j \\
\le\ & c_0 h\sum_{j=1}^{N} |p_j - \tilde{p}_j| + c_1 h \max_j |\tilde{p}_j| \left(N|q_i - \tilde{q}_i| + \sum_{j=1}^{N} |q_j - \tilde{q}_j| \right)
\end{aligned}$$

So if we denote $||v|| = \sum_{i=1}^{N} |v_i|$ for $v \in \mathbb{R}^N,\ L = \max\{c_0 h, c_1 h l_p N\}$, then we have the wanted estimate:

$$\left| h\sum_{j=1}^{N} G(q_i - q_j)p_j - h\sum_{j=1}^{N} G(\tilde{q}_i - \tilde{q}_j)\tilde{p}_j \right| \le L(||p - \tilde{p}|| + ||q - \tilde{q}||). \tag{6.18}$$

For the global in time existence we have

Proposition 6.2 *If the initial momenta are positive, $p_i \ge \varepsilon > 0,\ i = 1, 2, \cdots, N$, then the solution to (6.17) exists uniquely for all times.*

Proof From the Hamiltonian structure of (6.17), it is not difficult to prove that $P = h\sum_{i=1}^{N} p_i$ is independent of time t. And from the second equation of (6.17), we have $|\frac{\dot{p}_i}{p_i}| \le c_1 P$, where $c_1 = \max |G'(x)|$, so

$$p_i(0)e^{-c_1 P t} \le p_i(t) \le p_i(0)e^{c_1 P t}. \tag{6.19}$$

If the initial momenta are positive, then $p_i(t)$ are positive and bounded for all times $t < +\infty$. So the global existence follows.

Now we will prove that the solution of (6.17) converges to that of (6.16) as $h \to 0$.

Let $q(\xi, t), p(\xi, t)$ be the solution to (6.16) with the initial data

$$q(\xi, 0) = \xi, \quad p(\xi, 0) = p^0(\xi), \tag{6.20}$$

while $\tilde{q}(t),\ \tilde{p}(t)$ stand for the solution to (6.17) with

$$\tilde{q}_i(0) = q(\xi_i, 0) = \xi_i, \quad \tilde{p}_i(0) = p^0(\xi_i). \tag{6.21}$$

$q_i(t) = q(\xi_i, t)$ denotes the PDE solution evaluated at the grid points, $\phi_i = q_i - \tilde{q}_i, \psi_i = p_i - \tilde{p}_i$ and $||\phi|| = h\sum_{i=1}^N |\phi_i|,\ ||\psi|| = h\sum_{i=1}^N |\psi_i|$ denote the l_1 norm. From (6.15)(6.19) we easily know that for any $T > 0$, there exists a constant $P_T < \infty$, independent of h, such that

$$\max\{p_i(t) : 1 \le i \le N; 0 \le t \le T\}, \max\{p(\xi, t) : \xi \in S; 0 \le t \le T\} \le P_T$$

for h small enough (or equivalently, for N large enough). Denote $c_0 = \max|G(x)|,\ c_1 = \max|G'(x)|,\ c_2 = \max|G''(x)|$.

Theorem 6.1 *Consider (6.16) with (6.20) and (6.17) with (6.21). If* $p^0(\xi) > 0,\ \xi \in S$, *smooth enough, then for any finite time* $T > 0$, *there exists a grid length* h *such that*

$$||\phi(t)|| + \frac{1}{P_T}||\psi(t)|| \le \frac{Ch^2}{P_T}(e^{C'P_T t} - 1) \tag{6.22}$$

for $0 \le t \le T$, *where* $C,\ C'$ *are constants independent of* T *and* h.

Proof Because the two equations have the same initial values at the grid points, so

$$\begin{aligned}|\tilde{q}_i(t) - q(\xi_i, t)| \quad &\le \quad \int_0^t \left| h\sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s) - \int_S G(q(\xi_i, s) - q(\eta, s))p(\eta, s)\mathrm{d}\eta \right| \mathrm{d}s \\ &\quad + h\int_0^t \left| \sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))\tilde{p}_j(s) - \sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s) \right| \mathrm{d}s.\end{aligned} \tag{6.23}$$

The first integral of the right hand side is controlled above by Ch^2t because the Remannian sum $h\sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s)$ is the composite trapezoidal approximation of the integral $\int_S G(q(\xi_i, s) - q(\eta, s))p(\eta, s)\mathrm{d}\eta$. The second one is estimated as follows:

$$\begin{aligned} & h\left|\sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))\tilde{p}_j(s) - \sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s)\right| \\ \le\ & h\left|\sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))\tilde{p}_j(s) - \sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))p_j(s)\right| \\ & + h\left|\sum_{j=1}^N G(\tilde{q}_i(s) - \tilde{q}_j(s))p_j(s) - \sum_{j=1}^N G(q_i(s) - q_j(s))p_j(s)\right| \\ \le\ & c_0h\sum_{j=1}^N |p_j(s) - \tilde{p}_j(s)| + c_1h\sum_{j=1}^N \left(|q_i(s) - \tilde{q}_i(s)| + |q_j(s) - \tilde{q}_j(s)|\right) p_j(s) \\ \le\ & c_0h\sum_{j=1}^N |p_j(s) - \tilde{p}_j(s)| + c_1P_Th\sum_{j=1}^N \left(|q_i(s) - \tilde{q}_i(s)| + |q_j(s) - \tilde{q}_j(s)|\right), \end{aligned}$$

so

$$|\phi_i| \le \int_0^t \left(c_0h\sum_{j=1}^N |\psi_j| + c_1P_Th\sum_{j=1}^N (|\phi_i| + |\phi_j|) \right) \mathrm{d}s + Ch^2t, \tag{6.24}$$

and hence

$$||\phi(t)|| \leq \int_0^t \left(c_0||\psi|| + 2c_1 P_T ||\phi||\right) \mathrm{d}s + Ch^2 t. \tag{6.25}$$

Similarly, we have

$$|\psi_i(t)| \leq \int_0^t \left(2c_1 P_T h |p_i(s) - \tilde{p}_i(s)| + c_2 p_i P_T h \sum_{j=1}^{N} (|q_i(s) - \tilde{q}_i(s)| + |q_j(s) - \tilde{q}_j(s)|)\right) \mathrm{d}s + Ch^2 t, \tag{6.26}$$

and hence

$$||\psi(t)|| \leq \int_0^t \left(2c_1 h P_T ||\psi(s)|| + 2c_2 {P_T}^2 ||\phi(s)||\right) \mathrm{d}s + Ch^2 t, \tag{6.27}$$

from which we have

$$\begin{aligned} ||\phi(t)|| + \frac{1}{P_T}||\psi(t)|| &\leq \int_0^t (2(c_1+c_2)P_T||\phi|| + (c_0 + 2c_1 h)||\psi||) \,\mathrm{d}s + Ch^2 t \\ &= 2(c_1+c_2)P_T \int_0^t (||\phi|| + \frac{1}{P_T}||\psi||)\mathrm{d}s + (2c_1 h + c_0 - 2c_1 - 2c_2)\int_0^t ||\psi||\mathrm{d}s + Ch^2 t \\ &\leq 2(c_1+c_2)P_T \int_0^t (||\phi|| + \frac{1}{P_T}||\psi||)\mathrm{d}s + Ch^2 t, \end{aligned} \tag{6.28}$$

as long as $h < 1 + \frac{2c_2 - c_0}{2c_1}$! (It's easy to verify that $1 + \frac{2c_2 - c_0}{2c_1}$ is indeed a positive number.) Now (6.28) and Gronwall's inequality yield

$$||\phi(t)|| + \frac{1}{P_T}||\psi(t)|| \leq \frac{Ch^2}{2(c_1+c_2)P_T}(e^{2(c_1+c_2)P_T t} - 1) \tag{6.29}$$

for $0 < t < T$.

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