

Geodesic Warps by Conformal Mappings

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Abstract In recent years there has been considerable interest in methods for diffeomorphic warping of images, with applications in e.g. medical imaging and evolutionary biology. The original work generally cited is that of the evolutionary biologist D’Arcy Wentworth Thompson, who demonstrated warps to deform images of one species into another. However, unlike the deformations in modern methods, which are drawn from the full set of diffeomorphisms, he deliberately chose lower-dimensional sets of transformations, such as planar conformal mappings. In this paper we study warps composed of such conformal mappings. The approach is to equip the infinite dimensional manifold of conformal embeddings with a Riemannian metric, and then use the corresponding geodesic equation in order to obtain diffeomorphic warps. After deriving the geodesic equation, a numerical discretisation method is developed. Several examples of geodesic warps are then given. We also show that the equation admits totally geodesic solutions corresponding to scaling and translation, but not to affine transformations.

Keywords Image registration · Conformal mappings · Infinite dimensional manifolds · Geodesic warps · LDDMM

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1 Introduction

The use of diffeomorphic transformations in both image registration and shape analysis is now common, and utilised in many machine vision and image analysis tasks. One image or shape is brought into alignment with another by deforming the image until some similarity measure, such as sum-of-squares distance between pixels in the two images, reaches a minimum. The deformation is computed as a geodesic curve with respect to some metric on the diffeomorphism group. For a general treatment and an overview of the subject see the monograph by [Younes \(2010\)](#) and references therein.

The standard approach to the deformation method is to first perform an affine registration (principally to remove translation and rotation), and then to seek a geodesic warp of the image in the full set of diffeomorphisms of a fixed domain. Typically the setting is to use the right-invariant H_α^1 metric, which leads to the so-called EPDiff equation (see e.g. [Holm and Marsden 2005](#)). However, in what is arguably the most influential demonstration of the application of warping methods—the evolutionary biologist D’Arcy Wentworth Thompson’s (1942) seminal book ‘On Growth and Form’—Thompson warps images of one biological species into another using relatively simple types of transformation so that the gross features of the two images match. In a recent review of his work, biologist Arthur Wallace says:

This theory cries out for causal explanation, which is something the great man eschewed. [...] His transformations suggest coordinated rather than piecemeal changes to development in the course of evolution, an issue which almost completely disappeared from view in the era of the ‘modern synthesis’ of evolutionary theory, but which is of central importance again in the era of evo-devo. [...] All the tools are now in place

to examine the mechanistic basis of transformations. Not only do we have phylogenetic systematics and evo-devo, but, so obvious that it is easy to forget, we have computers, and especially, in this context, advanced computer graphics. We owe it to the great man to put these three things together to investigate the mechanisms that produce the morphological changes that he captured so elegantly with little more than sheets of graph paper and, of course, a brilliant mind (Wallace 2006).

We draw attention to two key aspects of Thompson's examples: (i) the transformations are as simple as possible to achieve what he considers a good enough match (see Table 1); and (ii) the classes of transformations that he considers all forms groups (or pseudogroups), either finite or infinite dimensional. Mostly, he uses conformal transformations, a constraint he is reluctant to give up:

It is true that, in a mathematical sense, it is not a perfectly satisfactory or perfectly regular deformation, for the system is no longer isogonal; but [. . .] approaches to an isogonal system under certain conditions of friction or constraint (Thompson 1942, p. 1064).
[. . .] it will perhaps be noticed that the correspondence is not always quite accurate in small details. It could easily have been made much more accurate by giving a slightly sinuous curvature to certain of the coordinates. But as they stand, the correspondence indicated is very close, and the simplicity of the figures illustrates all the better the general character of the transformation [ibid., p. 1074].

For applications in image registration we therefore suggest varying the group of transformations from which warps are drawn. If a low dimensional group gives a close match, then it should be preferred over a similar match from a higher-dimensional group. If necessary, local deformations from the full diffeomorphism group can be added later to account for fine details. In this paper we consider the case of conformal transformations. More precisely, we consider the problem of formulating and solving a geodesic equation on the space of conformal mappings. This is a fundamental sub-task in the framework of *large deformation diffeomorphic metric mapping* (LDDMM) (Trouvé 1995; Dupuis and Grenander 1998; Trouvé 1998; Joshi and Miller 2000; Miller and Younes 2001; Beg 2003; Beg et al. 2005; Bruveris et al. 2011), which is the standard setup for diffeomorphic image registration. Based on the geodesic equation derived in this paper, the full conformal image registration problem will be considered in future work.

Although the composition of two conformal maps is conformal, it need not be invertible: we need to restrict the

domain. The invertible conformal maps from the disk to itself do form a group, the disk-preserving Möbius group, but it is only three-dimensional. We are therefore led to consider the infinite dimensional configuration space $\text{Con}(\mathbb{U}, \mathbb{R}^2)$ of conformal embeddings of a simply connected compact domain $\mathbb{U} \subset \mathbb{R}^2$ into the plane. This set is not a group, but it is a pseudogroup.

In Marsland et al. (2011a) the authors study a geodesic equation using an L^2 -metric on the infinite dimensional manifold of conformal embeddings and a numerical method is developed for the initial value problem, based on the reproducing Bergman kernel. Using numerical examples, it is shown that the geodesic equation is ill-conditioned as an initial value problem, and that cusps are developed in finite time, which leads to a break-down of the dynamics.

In this paper we continue the study of geodesics on the manifold of conformal embeddings, but now with respect to a more general class of Sobolev type H_α^1 metrics. Furthermore, we develop a new numerical algorithm for solving the equations. The new method is based on a discrete variational principle, and directly solves the two point boundary value problem. Our new numerical method behaves well, i.e., converges fast, for all the examples we tried as long as the distance between the initial and final point on the manifold is not too large. From this observation we expect that the initial value problem with $\alpha > 0$ is well-posed in the H^s Banach space topology, which would imply that the Riemann exponential is a local diffeomorphism (see Lang 1999; Ebin and Marsden 1970; Shkoller 1998). This question will be investigated in detail in future work, as it is out of the scope of the current paper. The experimental results in this paper (Sect. 6) are limiting to confirming that the discrete Lagrangian method can reproduce the known geodesics consisting of linear conformal maps and can also calculate non-linear geodesics with moderately large deformations. This is a first step towards exploring the metric geometry of the conformal embeddings, similar to what was done for metrics on planar curves by Michor and Mumford (2006).

For analysis of 2D shapes, a setting using conformal mappings is developed in Sharon and Mumford (2005). There it is shown that the space of planar shapes is isomorphic to the quotient space $\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})$, where $\text{PSL}(2, \mathbb{R})$ acts on $\text{Diff}(S^1)$ by right composition of its corresponding disk-preserving Möbius transformation restricted to S^1 . Furthermore it is shown that there is a natural metric on $\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R})$, the Weil–Peterson metric, which has non-positive sectional curvature. The setting in this paper is related but different: rather than studying planar shapes we study conformal transformations between planar domains and we think of the manifold of conformal embeddings as a submanifold of the full space of planar embeddings. The equation we obtain can be seen as a generalisation and a restriction of the EPDiff equation. First, a generalisation by

Table 1 Transformation groups used in some transformations in Chap. XII, ‘On the Theory of Transformations, or the Comparison of Related Forms’, of [Thompson \(1942\)](#)

Figure no. in Thompson (1942)	Transformation group
515	$x \mapsto ax, y \mapsto y$
513.2	$x \mapsto ax, y \mapsto by$
509, 510, 518	$x \mapsto ax, y \mapsto cx + dy$ (shears)
521–522, 513.5	$x \mapsto ax + by, y \mapsto cx + dy$ (affine)
506, 508	$x \mapsto ax, y \mapsto g(y)$
511	$x \mapsto f(x), y \mapsto g(y)$
517–520, 523, 513.1, 513.3, 513.4, 513.6, 514, 525	Conformal
524	‘Peculiar’

going from the group of diffeomorphisms of a fixed domain to the manifold of embeddings from a planar domain into the entire plane.^{1,2} Second, a restriction by restricting to conformal embeddings. The approach we take is similar to (and much influenced by) the recent paper of [Gay-Balmaz et al. \(2012\)](#), in which a geometric framework for moving boundary continuum equations in physics is developed.

2 Mathematical Setting and Choice of Metric

The linear space of smooth maps $U \rightarrow \mathbb{R}^2$ is denoted $C^\infty(U, \mathbb{R}^2)$. Recall that this space is a Fréchet space, i.e., it has a topology defined by a countable set of semi-norms (see [Hamilton 1982](#), Sect. I.1 for details on the Fréchet topology used). The full set of embeddings $U \rightarrow \mathbb{R}^2$, denoted $\text{Emb}(U, \mathbb{R}^2)$, is an open subset of $C^\infty(U, \mathbb{R}^2)$. In particular, this implies that $\text{Emb}(U, \mathbb{R}^2)$ is a Fréchet manifold (see [Hamilton 1982](#), Sect. I.4.1).

Since $U \subset \mathbb{R}^2$ it holds that $\text{Emb}(U, \mathbb{R}^2)$ contains the identity mapping on U , which we denote by Id . The tangent space $T_{\text{Id}}\text{Emb}(U, \mathbb{R}^2)$ at the identity is given by the smooth vector fields on U , which we denote by $\mathfrak{X}(U)$. Notice that the vector fields need not be tangential to the boundary ∂U . Also notice that $\mathfrak{X}(U)$ is a Fréchet Lie algebra with bracket given by minus the Lie derivative on vector fields, i.e., if $\xi, \eta \in \mathfrak{X}(U)$, then $\text{ad}_\xi(\eta) = -\mathfrak{L}_\xi \eta$.

Let $\mathbf{g} = dx \otimes dx + dy \otimes dy$ be the standard Euclidean metric on \mathbb{R}^2 . Consider the subspace of $C^\infty(U, \mathbb{R}^2)$ consisting of maps that preserve the metric up to multiplication with elements in the space $\mathcal{F}(U)$ of smooth real valued functions on U . That is, the subspace

¹ This generalisation of EPDiff has not yet been worked out in detail in the literature. However, it is likely that the approach developed in [Gay-Balmaz et al. \(2012\)](#) for free boundary flow can be used with only minor modifications.

² One can also look at the generalisation of EPDiff to embeddings from a Klein geometry perspective. Indeed, let $\text{Diff}_U(\mathbb{R}^2)$ denote the diffeomorphisms that leaves the domain U invariant. Then the embeddings $\text{Emb}(U, \mathbb{R}^2)$ can be identified with the space of co-sets $\text{Diff}(\mathbb{R}^2)/\text{Diff}_U(\mathbb{R}^2)$.

$$C_c^\infty(U, \mathbb{R}^2) = \{\phi \in C^\infty(U, \mathbb{R}^2); \phi^* \mathbf{g} = F\mathbf{g}, F \in \mathcal{F}(U)\}.$$

This subspace is topologically closed in $C^\infty(U, \mathbb{R}^2)$. The set of conformal embeddings $\text{Con}(U, \mathbb{R}^2) = \text{Emb}(U, \mathbb{R}^2) \cap C_c^\infty(U, \mathbb{R}^2)$ is an open subset of $C_c^\infty(U, \mathbb{R}^2)$ and a Fréchet submanifold of $\text{Emb}(U, \mathbb{R}^2)$. The tangent space $T_{\text{Id}}\text{Con}(U, \mathbb{R}^2)$ is given by

$$\mathfrak{X}_c(U) = \{\xi \in \mathfrak{X}(U); \mathfrak{L}_\xi \mathbf{g} = \text{div}(\xi)\mathbf{g}\},$$

which follows by straightforward calculations. Notice that $\mathfrak{X}_c(U)$ is a subalgebra of $\mathfrak{X}(U)$, since

$$\begin{aligned} \mathfrak{L}_{\mathfrak{L}_{\xi\eta}} \mathbf{g} &= \mathfrak{L}_\xi \mathfrak{L}_\eta \mathbf{g} - \mathfrak{L}_\eta \mathfrak{L}_\xi \mathbf{g} = \mathfrak{L}_\xi(\text{div}(\eta)\mathbf{g}) - \mathfrak{L}_\eta(\text{div}(\xi)\mathbf{g}) \\ &= (\mathfrak{L}_\xi \text{div}(\eta) - \mathfrak{L}_\eta \text{div}(\xi))\mathbf{g} + \underbrace{\text{div}(\eta)\mathfrak{L}_\xi \mathbf{g} - \text{div}(\xi)\mathfrak{L}_\eta \mathbf{g}}_0 \\ &= \text{div}(\mathfrak{L}_{\xi\eta})\mathbf{g}. \end{aligned}$$

In the forthcoming, we identify the plane \mathbb{R}^2 with the complex numbers \mathbb{C} through $(x, y) \mapsto z = x + iy$. Hence, the vector fields $\mathfrak{X}(U)$ are identified with smooth complex valued functions on U , and $\mathfrak{X}_c(U)$ with the space of holomorphic functions.

The complex L^2 inner product on $\mathfrak{X}(U)$ is given by

$$\langle\langle \xi, \eta \rangle\rangle_{L^2(U)} := \int_U \xi(z) \overline{\eta(z)} dA(z),$$

where $dA = dx \wedge dy$ is the canonical volume form on \mathbb{R}^2 . Correspondingly, we also have the real L^2 inner product given by

$$\langle \xi, \eta \rangle_{L^2(U)} := \int_U \mathbf{g}(\xi, \eta) dA = \text{Re} \langle\langle \xi, \eta \rangle\rangle_{L^2(U)}.$$

Also, we have the more general class of real and complex H_α^1 inner products given by

$$\begin{aligned} \langle \xi, \eta \rangle_{H_\alpha^1(\mathbf{U})} &:= \langle \xi, \eta \rangle_{L^2(\mathbf{U})} + \frac{\alpha}{2} \langle \xi_x, \eta_x \rangle_{L^2(\mathbf{U})} \\ &\quad + \frac{\alpha}{2} \langle \xi_y, \eta_y \rangle_{L^2(\mathbf{U})}, \\ \langle \langle \xi, \eta \rangle \rangle_{H_\alpha^1(\mathbf{U})} &:= \langle \langle \xi, \eta \rangle \rangle_{L^2(\mathbf{U})} + \frac{\alpha}{2} \langle \langle \xi_x, \eta_x \rangle \rangle_{L^2(\mathbf{U})} \\ &\quad + \frac{\alpha}{2} \langle \langle \xi_y, \eta_y \rangle \rangle_{L^2(\mathbf{U})}, \end{aligned}$$

where $\alpha \geq 0$ and ξ_x, ξ_y respectively denotes derivatives with respect to the Cartesian coordinates (x, y) . Notice that if $\xi, \eta \in \mathfrak{X}_c(\mathbf{U})$, then

$$\langle \langle \xi, \eta \rangle \rangle_{H_\alpha^1(\mathbf{U})} = \langle \langle \xi, \eta \rangle \rangle_{L^2(\mathbf{U})} + \alpha \langle \langle \xi', \eta' \rangle \rangle_{L^2(\mathbf{U})}$$

where ξ' and η' denote complex derivatives.

The class of inner products $\langle \cdot, \cdot \rangle_{H_\alpha^1(\mathbf{U})}$ on $T_{\text{Id}}\text{Emb}(\mathbf{U}, \mathbb{R}^2) = \mathfrak{X}(\mathbf{U})$ induces a corresponding class of Riemannian metrics on $\text{Emb}(\mathbf{U}, \mathbb{R}^2)$ by

$$\begin{aligned} T_\varphi\text{Emb}(\mathbf{U}, \mathbb{R}^2) \times T_\varphi\text{Emb}(\mathbf{U}, \mathbb{R}^2) \ni (U, V) &\mapsto \\ \langle U \circ \varphi^{-1}, V \circ \varphi^{-1} \rangle_{H_\alpha^1(\varphi(\mathbf{U}))}. &\quad (1) \end{aligned}$$

Note that φ^{-1} is well-defined as a map $\varphi(\mathbf{U}) \rightarrow \mathbf{U}$ since φ is an embedding. Also note that the restriction of the metric (1) to the submanifold of diffeomorphisms $\text{Diff}(\mathbf{U}) \subset \text{Emb}(\mathbf{U}, \mathbb{R}^2)$ yields the ‘‘ordinary’’ H_α^1 metric on $\text{Diff}(\mathbf{U})$ corresponding to the EPDiff equations.

3 Derivation of the Geodesic Equation

In this section we derive the geodesic equation on $\text{Con}(\mathbf{U}, \mathbb{R}^2)$ for the class of $H_\alpha^1(\mathbf{U})$ metrics given by (1). These equations are given by the Euler–Lagrange equations with respect to the quadratic Lagrangian on $\text{Con}(\mathbf{U}, \mathbb{R}^2)$ given by

$$\begin{aligned} L(\varphi, \dot{\varphi}) &= \frac{1}{2} \langle \dot{\varphi} \circ \varphi^{-1}, \dot{\varphi} \circ \varphi^{-1} \rangle_{H_\alpha^1(\varphi(\mathbf{U}))} \\ &= \frac{1}{2} \langle \langle \dot{\varphi} \circ \varphi^{-1}, \dot{\varphi} \circ \varphi^{-1} \rangle \rangle_{H_\alpha^1(\varphi(\mathbf{U}))}, \end{aligned} \quad (2)$$

where $\dot{\varphi} \in T_\varphi\text{Emb}(\mathbf{U}, \mathbb{R}^2)$ corresponds to the time derivative.

As a first step, we have the following result.

Lemma 1 *For any $(\varphi, \dot{\varphi}) \in T\text{Emb}(\mathbf{U}, \mathbb{R}^2)$ it holds that*

$$\begin{aligned} \langle \dot{\varphi} \circ \varphi^{-1}, \dot{\varphi} \circ \varphi^{-1} \rangle_{H_\alpha^1(\varphi(\mathbf{U}))} &= \langle \langle \dot{\varphi}' \dot{\varphi}, \dot{\varphi}' \dot{\varphi} \rangle \rangle_{L^2(\mathbf{U})} \\ &\quad + \alpha \langle \langle \dot{\varphi}', \dot{\varphi}' \rangle \rangle_{L^2(\mathbf{U})}. \end{aligned}$$

Proof Let f be any complex valued function on \mathbf{U} . By a change of variables $w = \varphi(z)$ we obtain

$$\int_{\varphi(\mathbf{U})} f \circ \varphi^{-1}(w) dA(w) = \int_{\mathbf{U}} f(z) |\varphi'(z)|^2 dA(z).$$

For the first term we take $f(z) = \dot{\varphi}(z) \overline{\dot{\varphi}(z)}$ which yields

$$\begin{aligned} \langle \dot{\varphi} \circ \varphi^{-1}, \dot{\varphi} \circ \varphi^{-1} \rangle_{L^2(\varphi(\mathbf{U}))} &= \int_{\mathbf{U}} \dot{\varphi}(z) \overline{\dot{\varphi}(z)} |\varphi'(z)|^2 dA(z) \\ &= \langle \langle \dot{\varphi}' \dot{\varphi}, \dot{\varphi}' \dot{\varphi} \rangle \rangle_{L^2(\mathbf{U})}. \end{aligned}$$

For the second term we take first notice that

$$(\dot{\varphi} \circ \varphi^{-1})'(w) = \frac{\dot{\varphi}' \circ \varphi^{-1}(w)}{\varphi' \circ \varphi^{-1}(w)}$$

and then we take $f(z) = \dot{\varphi}'(z)/\varphi'(z)$. The result now follows. \square

We are now ready to derive the Euler–Lagrange equations from the variational principle. We look for a curve $\varphi : [0, 1] \rightarrow \text{Con}(\mathbf{U}, \mathbb{R}^2)$ such that

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_0^1 \frac{1}{2} \langle \langle \dot{\varphi}_\varepsilon(t) \circ \varphi_\varepsilon(t)^{-1}, \dot{\varphi}_\varepsilon(t) \circ \varphi_\varepsilon(t)^{-1} \rangle \rangle_{H_\alpha^1(\varphi(\mathbf{U}))} \\ dt = 0 \end{aligned}$$

for all variations $\varphi_\varepsilon : [0, 1] \rightarrow \text{Con}(\mathbf{U}, \mathbb{R}^2)$ such that $\varphi_\varepsilon(0) = \varphi(0)$, $\varphi_\varepsilon(1) = \varphi(1)$ and $\varphi_0 = \varphi$. To simplify notation we introduce

$$\psi = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \varphi_\varepsilon.$$

Notice that $\psi(0) = \psi(1) = 0$. Using Lemma 1 and the fact that differentiation commutes with integration we obtain

$$\begin{aligned} 0 &= \int_0^1 \left(\langle \langle \dot{\varphi}' \dot{\varphi}, \left. \frac{d}{d\varepsilon} \right|_{\varepsilon} \varphi'_\varepsilon \dot{\varphi}_\varepsilon \rangle \rangle_{L^2(\mathbf{U})} + \alpha \langle \langle \dot{\varphi}', \left. \frac{d}{d\varepsilon} \right|_{\varepsilon} \dot{\varphi}'_\varepsilon \rangle \rangle_{L^2(\mathbf{U})} \right) dt \\ &= \int_0^1 \left(\langle \langle \dot{\varphi}' \dot{\varphi}, \psi' \dot{\varphi} + \dot{\varphi}' \psi \rangle \rangle_{L^2(\mathbf{U})} + \alpha \langle \langle \dot{\varphi}', \psi' \rangle \rangle_{L^2(\mathbf{U})} \right) dt \\ &= \int_0^1 \left(\langle \langle \dot{\varphi}' \dot{\varphi}, \psi' \dot{\varphi} - \dot{\varphi}' \psi + \frac{d}{dt}(\varphi' \psi) \rangle \rangle_{L^2(\mathbf{U})} - \alpha \langle \langle \ddot{\varphi}', \psi' \rangle \rangle_{L^2(\mathbf{U})} \right) dt \\ &= \int_0^1 \left(- \langle \langle \frac{d}{dt}(\varphi' \dot{\varphi}), \varphi' \psi \rangle \rangle_{L^2(\mathbf{U})} \right. \\ &\quad \left. + \langle \langle \dot{\varphi}' \dot{\varphi}, \psi' \dot{\varphi} - \dot{\varphi}' \psi \rangle \rangle_{L^2(\mathbf{U})} - \alpha \langle \langle \ddot{\varphi}', \psi' \rangle \rangle_{L^2(\mathbf{U})} \right) dt, \end{aligned}$$

where in the last two equalities we use integration by parts over the time variable and the fact that ψ vanishes at the endpoints. Notice that there are now no time derivatives on ψ . Thus, by the fundamental lemma of calculus of variations we can remove the time integration and thereby obtain a weak equation which must be fulfilled at each point in time. In order to obtain a strong formulation, we also need to isolate ψ from spatial derivatives. The standard approach of using integration by parts introduces a boundary integral term.

In most examples of calculus of variations, this boundary term either vanishes (in the case of a space of tangential vector fields), or can be treated separately, giving rise to natural boundary conditions (in the case of a space where vector fields can have arbitrary small compact support). However, in the case of conformal mappings, there is always a global dependence between interior points, and points on the boundary (since holomorphic functions cannot have local support). Hence, we need an appropriate analogue of integration by parts that avoids boundary integrals. For this, consider the adjoint operator of complex differentiation, i.e., an operator $\partial_z^\top : \mathfrak{X}_c(\mathbf{U}) \rightarrow \mathfrak{X}_c(\mathbf{U})$ such that

$$\langle\langle \xi, \eta' \rangle\rangle_{L^2(\mathbf{U})} = \langle\langle \partial_z^\top \xi, \eta \rangle\rangle_{L^2(\mathbf{U})}, \quad \forall \xi, \eta \in \mathfrak{X}_c(\mathbf{U}).$$

Notice that ∂_z^\top depends on the domain \mathbf{U} . In the case of the unit disk $\mathbf{U} = \mathbb{D}$, it holds that $\partial_z^\top \xi(z) = \partial_z(z^2 \xi'(z)) = 2z \xi'(z) + z^2 \xi''(z)$. In the general case, this operator is more complicated, but can still be computed under the assumption that a conformal embedding $\mathbf{U} \rightarrow \mathbb{D}$ is known (see Marsland et al. 2011b).

Using the operator ∂_z^\top we can now proceed as follows

$$\begin{aligned} 0 &= -\langle\langle \frac{d}{dt}(\varphi' \dot{\varphi}), \varphi' \psi \rangle\rangle_{L^2(\mathbf{U})} + \langle\langle \varphi' \dot{\varphi}, \psi' \dot{\varphi} - \dot{\varphi}' \psi \rangle\rangle_{L^2(\mathbf{U})} \\ &\quad - \alpha \langle\langle \ddot{\varphi}', \psi' \rangle\rangle_{L^2(\mathbf{U})} \\ &= -\langle\langle |\varphi'|^2 \ddot{\varphi} + \dot{\varphi} \dot{\varphi}' \overline{\varphi'}, \psi \rangle\rangle_{L^2(\mathbf{U})} - \langle\langle \dot{\varphi} \varphi' \overline{\varphi}', \psi \rangle\rangle_{L^2(\mathbf{U})} \\ &\quad + \langle\langle \varphi' \dot{\varphi}, (\psi \dot{\varphi})' - \psi \dot{\varphi}' \rangle\rangle_{L^2(\mathbf{U})} - \alpha \langle\langle \partial_z^\top \ddot{\varphi}', \psi \rangle\rangle_{L^2(\mathbf{U})} \\ &= -\langle\langle |\varphi'|^2 \ddot{\varphi} + \dot{\varphi} \dot{\varphi}' \overline{\varphi'} + \dot{\varphi} \varphi' \overline{\varphi}' + \alpha \partial_z^\top \ddot{\varphi}', \psi \rangle\rangle_{L^2(\mathbf{U})} \\ &\quad + \langle\langle \partial_z^\top (\varphi' \dot{\varphi}), \psi \dot{\varphi} \rangle\rangle_{L^2(\mathbf{U})} \\ &= -\langle\langle (|\varphi'|^2 + \alpha \partial_z^\top \partial_z) \ddot{\varphi} + \dot{\varphi} (\dot{\varphi}' \overline{\varphi'} + \varphi' \overline{\varphi}'') \rangle\rangle_{L^2(\mathbf{U})} \\ &\quad - \langle\langle \overline{\varphi} \partial_z^\top (\varphi' \dot{\varphi}), \psi \rangle\rangle_{L^2(\mathbf{U})}. \end{aligned}$$

Thus, this relation must hold for all holomorphic functions ψ . However, the expression in the first slot of the inner product is not holomorphic, so it needs to be orthogonally projected back to the set of holomorphic functions. Using Hodge theory for manifolds with boundary, one can show that the orthogonal complement of $\mathfrak{X}_c(\mathbf{U})$ in $\mathfrak{X}(\mathbf{U})$ with respect to the real inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbf{U})}$ is given by

$$\begin{aligned} \mathfrak{X}_c(\mathbf{U})^\perp &= \{ \xi \in \mathfrak{X}(\mathbf{U}); \xi(z) \\ &= \partial_x F - i \partial_y F + \partial_y G + i \partial_x G, \quad F, G \in \mathcal{F}_0(\mathbf{U}) \}, \end{aligned}$$

where $\mathcal{F}_0(\mathbf{U}) = \{ F \in \mathcal{F}(\mathbf{U}); F|_{\partial \mathbf{U}} = 0 \}$ are the smooth functions that vanish at the boundary. This result is obtained in Marsland et al. (2011b). Since $\mathfrak{X}_c(\mathbf{U})^\perp$ is invariant under multiplication with i , it is also the orthogonal complement with respect to the complex L^2 inner product.

Now we finally arrive at the strong formulation of the Euler–Lagrange equations

$$\begin{aligned} \frac{d}{dt} (\mathcal{A}(\varphi) \dot{\varphi}) - \overline{\varphi} \partial_z^\top (\varphi' \dot{\varphi}) &= \mathcal{A}(\varphi) (\partial_x F - i \partial_y F \\ &\quad + \partial_y G + i \partial_x G), \quad (3) \\ \partial_z \varphi &= 0, \\ F|_{\partial \mathbf{U}} = G|_{\partial \mathbf{U}} &= 0, \end{aligned}$$

where $\mathcal{A}(\varphi) = |\varphi'|^2 + \alpha \partial_z^\top \partial_z$ is the *inertia operator* (self adjoint with respect to the L^2 inner product) and where the second equation means that φ is constrained to be holomorphic. Indeed, one may think of Eq. (3) as a Lagrange–D’Alembert equation for a system with configuration space $\text{Emb}(\mathbf{U}, \mathbb{R}^2)$ which, by Lagrangian multipliers (F, G) , is constrained to the submanifold $\text{Con}(\mathbf{U}, \mathbb{R}^2)$.

In the special case $\mathbf{U} = \mathbb{D}$ we get

$$\begin{aligned} \frac{d}{dt} (\mathcal{A}(\varphi) \dot{\varphi}) - \overline{\varphi} \partial_z (z^2 \varphi' \dot{\varphi}) &= \mathcal{A}(\varphi) (\partial_x F - i \partial_y F \\ &\quad + \partial_y G + i \partial_x G), \quad (4) \\ \partial_z \varphi &= 0, \\ F|_{\partial \mathbf{U}} = G|_{\partial \mathbf{U}} &= 0. \end{aligned}$$

3.1 Weak Geodesic Equation in the Right Reduced Variable

It is also possible to derive the geodesic equation using the right reduced variable $\xi = \dot{\varphi} \circ \varphi^{-1}$, as is typically done for geodesic equations on diffeomorphism groups with invariant metric (see e.g. Arnold and Khesin 1998; Khesin and Wendt 2009; Modin et al. 2011). However, there is a difference between the setting of embeddings and that of diffeomorphism groups, since ξ is defined on $\varphi(\mathbf{U})$, which is not fixed in the embedding setting. Nevertheless, the “moving domain” $\Sigma = \varphi(\mathbf{U})$ simply moves along the flow, i.e., points on the boundary follows the flow of the vector field ξ . For details of this setting in the two cases of unconstrained embeddings and volume preserving embeddings, see Gay-Balmaz et al. (2012).

Let γ_ε be a variation of γ as above, and let $\xi_\varepsilon = \dot{\gamma}_\varepsilon \circ \gamma_\varepsilon^{-1}$. Using the calculus of Lie derivatives, direct calculations yield

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \xi_\varepsilon &= \dot{\eta} + \mathfrak{L}_\eta \xi \\ \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \frac{1}{2} \langle \xi, \xi \rangle_{H^1_\alpha(\gamma_\varepsilon(\mathbf{U}))} &= \langle \xi, \mathfrak{L}_\eta \xi \rangle_{L^2(\gamma(\mathbf{U}))} \\ &\quad + \alpha \langle \xi', \mathfrak{L}_\eta \xi' \rangle_{L^2(\gamma(\mathbf{U}))} \\ &\quad + \langle \xi, \text{div}(\eta) \xi \rangle_{L^2(\gamma(\mathbf{U}))} \\ &\quad + \alpha \langle \xi', \text{div}(\eta) \xi' \rangle_{L^2(\gamma(\mathbf{U}))} \end{aligned}$$

where $\eta = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \varphi_\varepsilon \circ \varphi^{-1}$. From the variational principle

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_0^1 L(\varphi_\varepsilon, \dot{\varphi}_\varepsilon) dt = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_0^1 \frac{1}{2} \langle \dot{\xi}_\varepsilon, \dot{\xi}_\varepsilon \rangle_{H^1_\alpha(\gamma_\varepsilon(U))} dt = 0$$

we now obtain the weak form of the equation in terms of the variables (φ, ξ) as

$$\int_0^1 \left(\langle \xi, \dot{\eta} + 2\xi_\eta \xi + \operatorname{div}(\eta)\xi \rangle_{L^2(\gamma(U))} + \alpha \langle \xi', \dot{\eta}' + \partial_z \xi_\eta \xi + \xi_\eta \xi' + \operatorname{div}(\eta)\xi' \rangle_{L^2(\gamma(U))} \right) dt = 0.$$

Passing now to the complex inner product, we use the formulas

$$\begin{aligned} \langle \xi, \eta \rangle_{L^2(U)} &= \langle \xi, \eta \rangle_{L^2(U)} + i \langle \xi, i\eta \rangle_{L^2(U)} \\ \mathfrak{g}(\xi, \operatorname{div}(\eta)\xi) + i\mathfrak{g}(\xi, \operatorname{div}(i\eta)\xi) &= 2\xi \overline{\eta' \xi} \\ \xi_{i\eta} \xi &= i\xi_\eta \xi \end{aligned}$$

which yields the weak equation

$$\int_0^1 \left(\langle \xi, \dot{\eta} + 2\xi_\eta \xi + 2\eta' \xi \rangle_{L^2(\gamma(U))} + \alpha \langle \xi', \dot{\eta}' + \partial_z(\eta' \xi) + 2\xi_\eta \xi' \rangle_{L^2(\gamma(U))} \right) dt = 0. \tag{5}$$

Together with the equation $\dot{\varphi} = \xi \circ \varphi$, this is thus a weak form of Eq. (3), but expressed in the variables (φ, ξ) .

4 Totally Geodesic Submanifolds

In this section we investigate special solutions to Eq. (4). The approach for doing so is to find a finite dimensional submanifold of $\operatorname{Con}(\mathbb{D}, \mathbb{R}^2)$ such that solutions curves starting and ending on this submanifold actually lie on the submanifold.

Recall that a submanifold $N \subset M$ of a Riemannian manifold (M, \mathfrak{g}) is *totally geodesic with respect to* (M, \mathfrak{g}) if geodesics in N (with respect to \mathfrak{g} restricted to N) are also geodesics in M . For a thorough treatment of totally geodesic subgroups of $\operatorname{Diff}(M)$ with respect to various metrics, see Modin et al. (2011).

Consider now the submanifold of linear conformal transformations

$$\operatorname{Lin}(\mathbb{D}, \mathbb{R}^2) = \left\{ \varphi \in \operatorname{Con}(\mathbb{D}, \mathbb{R}^2); \varphi(z) = cz, c \in \mathbb{C} \right\}.$$

Proposition 1 *Lin*(\mathbb{D}, \mathbb{C}) is totally geodesic in *Con*(\mathbb{D}, \mathbb{C}) with respect to the H^1_α metric given by (1).

Proof If $t \mapsto \varphi(t)$ is a path in *Lin*(\mathbb{D}, \mathbb{C}), i.e., $\varphi(z) = cz$ with $c \in \mathbb{C}$, then $\xi = \dot{\varphi} \circ \varphi^{-1}$ is of the form $\xi(z) = az$ with $a \in \mathbb{C}$. Now, let $t \mapsto (\varphi, \xi)$ fulfill the variational Eq. (5) for

each variation of the form $\eta(z) = bz$ with $b \in \mathbb{C}$. We need to show that $t \mapsto (\varphi, \xi)$ then fulfills the equation for any variation of the form $\eta(z) = z^k$ (since the monomials span the space of holomorphic functions). Thus, for the first term in (5) we get

$$\begin{aligned} &\langle \xi, \dot{\eta} + 2\xi_\eta \xi + 2\eta' \xi \rangle_{L^2(\varphi(\mathbb{D}))} \\ &= \langle \xi, \dot{\eta} + 4\eta' \xi - 2\xi' \eta \rangle_{L^2(\varphi(\mathbb{D}))} \\ &= \langle \varphi' \cdot \xi \circ \varphi, \varphi' \cdot (\dot{\eta} + 4\eta' \xi - 2\xi' \eta) \circ \varphi \rangle_{L^2(\mathbb{D})} \\ &= |c|^4 \langle az, \dot{b}z^k + 4kbaz^k - 2abz^k \rangle_{L^2(\mathbb{D})}. \end{aligned} \tag{6}$$

where in the first line we use the conformal change of variables formula for integrals. Now, since the monomials are orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{D}}$ the expression vanish whenever $k \neq 1$. For the second term, notice that $\xi' = a$ is constant. Now, if η is constant, then all the terms $\dot{\eta}', \eta' \xi, \xi_\eta \xi'$ vanish. If $\eta = z^k$ with $k \geq 2$, then the second term $\alpha \langle \xi', \dot{\eta}' + \partial_z(\xi \eta') + 2\xi_\eta \xi' \rangle$ is of the form

$$\alpha \int_{\mathbb{C}\mathbb{D}} \overline{a(b + 3ab)kz^{k-1}} dA(z) = \alpha |c|^2 \overline{a(b + 3ab)} \int_{\mathbb{D}} kz^{k-1}$$

which vanishes for every $a, c_1 \in \mathbb{C}$. This concludes the proof. \square

We now derive a differential equation for the totally geodesic solutions in *Lin*(\mathbb{D}, \mathbb{C}) in term of the variables (c, a) corresponding to $\varphi(z) = cz$ and $\xi(z) = az$. From $\dot{\varphi} = \xi \circ \varphi$ it follows that $\dot{c} = ac$. Next, we plug the ansatz into the weak Eq. (5), and use that b vanishes at the endpoints, which yields the equations

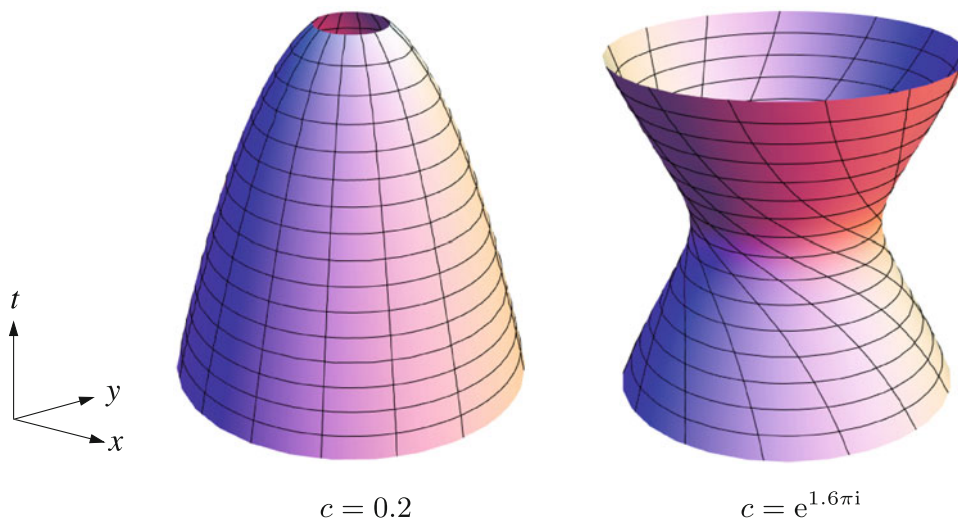
$$\begin{aligned} \dot{c} &= ac \\ \dot{a}(2c + \alpha) &= -4a^2c - \alpha a^2 \end{aligned} \tag{7}$$

These equations thus give special solutions to Eq. (4). We obtain that $\frac{d^2}{dt^2}(c^2 + \alpha c) = 0$, so we can analytically compute these special solutions. Notice that if a and c are initially real, then both a and c stay real, so the even smaller submanifold of pure scalings is also totally geodesic.

Figure 1 gives a visualization of total geodesic solutions where the two end transformations are given first by a pure scaling and second by a pure rotation. Notice that within the submanifold *Lin*(\mathbb{D}, \mathbb{C}), the smaller submanifold of scalings is totally geodesic, as is shown in the left figure. However, the submanifold of rotations is not, as is shown in the right figure.

Remark 1 By using again the weak form (5) of the governing equation one can further show that the submanifold of translations is *not* totally geodesic in *Con*(\mathbb{D}, \mathbb{C}). Nor is the submanifold of affine conformal transformations.

Fig. 1 Geodesic curve from $\varphi_0(z) = z$ to $\varphi_1(z) = cz$ for different values of c and $\alpha = 0$. The mesh lines show how the unit circle evolves. Notice that the scaling geodesic stays a scaling (*left figure*), whereas the rotation geodesic picks up some scaling during its time evolution (*right figure*)



5 Numerical Discretization

In this section we describe a method for numerical discretization of Eq. (4). The basic idea is to obtain a spatial discretization of the phase space variables $(\varphi, \dot{\varphi}) \in T\text{Con}(\mathbb{D}, \mathbb{R}^2)$ by truncation of the Taylor series. Thus, we use a Galerkin type approach for spatial discretization. For time discretization we take a variational approach, using the framework of discrete mechanics (cf. Marsden and West 2001).

The discrete configuration space is given by

$$Q_n = \{\varphi \in \text{Con}(\mathbb{D}, \mathbb{R}^2); \varphi(z) = \sum_{i=0}^{n-1} c_i z^i, c_i \in \mathbb{C}\}.$$

Notice that Q_n is an n -dimensional submanifold of $\text{Con}(\mathbb{D}, \mathbb{R}^2)$. Since $\text{Con}(\mathbb{D}, \mathbb{R}^2)$ is an open subset of the vector space of all holomorphic maps on \mathbb{D} (in the Fréchet topology), it holds that the discrete configuration space Q_n is an open subset of $\text{span}_{\mathbb{C}}\{1, z, \dots, z^{n-1}\} \simeq \mathbb{C}^n$. Thus, each tangent space $T_{\varphi}Q_n$ is identified with \mathbb{C}^n by taking the coefficients of the finite Taylor series. Together with the restricted H^1_{α} metric, Q_n is a Riemannian manifold.

Let $U, V \in T_{\varphi}\text{Con}(\mathbb{D}, \mathbb{R}^2)$, and let $(a_k)_{k=0}^{\infty}, (b_k)_{k=0}^{\infty}$ respectively be their Taylor coefficients. Then it holds that

$$\langle\langle U, V \rangle\rangle_{L^2(\mathbb{D})} = \frac{1}{\pi} \sum_{k=0}^{\infty} (1+i)a_k \bar{b}_k,$$

which follows since $\langle\langle z^i, z^j \rangle\rangle_{L^2(\mathbb{D})} = \delta_{ij}(i+1)/\pi$.

The next step is to obtain a numerical method that approximates the geodesics. For this we use the variational method obtained by the discrete Lagrangian on $Q_n \times Q_n$ given by

$$\begin{aligned} L_d(\varphi_k, \varphi_{k+1}) &= hL\left(\underbrace{\frac{\varphi_k + \varphi_{k+1}}{2}}_{\varphi_{k+1/2}}, \frac{\varphi_{k+1} - \varphi_k}{h}\right) \\ &= \frac{1}{2h} \langle\langle \varphi'_{k+1/2}(\varphi_{k+1} - \varphi_k), \varphi'_{k+1/2}(\varphi_{k+1} - \varphi_k) \rangle\rangle_{L^2(\mathbb{D})} \\ &\quad + \frac{\alpha}{2h} \langle\langle \varphi'_{k+1} - \varphi'_k, \varphi'_{k+1} - \varphi'_k \rangle\rangle_{L^2(\mathbb{D})}, \end{aligned}$$

where $h > 0$ is the step size. The discrete action is thus given by

$$A_d(\varphi_0, \dots, \varphi_N) = \sum_{k=0}^{N-1} L_d(\varphi_k, \varphi_{k+1}). \tag{8}$$

Now, a method for numerical computation of geodesics originating from the identity and ending at a known configuration is obtained as follows.

Algorithm 1 Given $\varphi \in \text{Con}(\mathbb{D}, \mathbb{R}^2)$, an approximation to the geodesic curve from the identity element $\varphi_0(z) = z$ to φ is given by the following algorithm:

1. Set $\varphi_N = \mathcal{P}_n\varphi$, where $\mathcal{P}_n : \text{Con}(\mathbb{D}, \mathbb{R}^2) \rightarrow Q_n$ is projection by truncation of Taylor series.
2. Set initial guess $\varphi_k = (1 - k/N)\varphi_0 + k/N\varphi_N$ for $k = 1, \dots, N - 1$.
3. Solve the minimization problem

$$\min_{\varphi_1, \dots, \varphi_{N-1} \in Q_n} A_d(\varphi_0, \dots, \varphi_N)$$

with a numerical non-linear numerical minimization algorithm.

Remark 2 In practical computations we use \mathbb{C}^n instead of Q_n . Thus, as a last step one have to check that the solution obtained fulfils that $\varphi_k \in Q_n$, i.e., $\varphi'_k(z) \neq 0$ for $z \in \mathbb{D}$.

For short enough geodesics, this is guaranteed by the fact that $\varphi'_0(z) = 1$.

Remark 3 Notice that we solve the problem as a two point boundary value problem. Thus, we assume that the final state φ_N is known. In future work we will consider the more general optimal control problem, where the final configuration is determined by minimizing a functional, such as sum-of-squares of the pixel difference between the destination and target image.

5.1 Efficient Evaluation of the Discrete Action

Evaluation of the discrete action functional (8) requires computation of the Taylor coefficients of the product $\varphi'_{k+1/2}(\varphi_{k+1} - \varphi_k)$. The “brute force” algorithm for doing this requires $\mathcal{O}(n^2)$ operations. However, it is well known that FFT techniques can be used to accelerate such computations. Using this, we now give an $\mathcal{O}(Nn \log n)$ algorithm for evaluation of the discrete action.

Algorithm 2 Given $\varphi_0, \dots, \varphi_N$, an efficient algorithm for computing the discrete action $A_d(\varphi_0, \dots, \varphi_N)$ is given by:

1. For each $k = 0, \dots, N$, compute the Taylor coefficients of φ'_k . This requires $\mathcal{O}(Nn)$ operations.

2. Compute

$$A_1 = \sum_{k=0}^{N-1} \frac{\alpha}{2h} \langle\langle \varphi'_{k+1} - \varphi'_k, \varphi'_{k+1} - \varphi'_k \rangle\rangle_{L^2(\mathbb{D})}.$$

This requires $\mathcal{O}(Nn)$ operations.

3. Set $\mathbf{a}_k \in \mathbb{C}^{2n}$ to contain the Taylor coefficients of $\varphi'_{k+1/2}$ as its first $n - 1$ elements, and then zero padded. This requires $\mathcal{O}(Nn)$ operations.
4. Set $\mathbf{b}_k \in \mathbb{C}^{2n}$ to contain the Taylor coefficients of $(\varphi_{k+1} - \varphi_k)/h$ as its first n elements, and then zero padded. This requires $\mathcal{O}(Nn)$ operations.
5. Compute

$$\hat{\mathbf{a}}_k = \text{FFT}(\mathbf{a}_k), \quad \hat{\mathbf{b}}_k = \text{FFT}(\mathbf{b}_k).$$

This requires $\mathcal{O}(Nn \log n)$ operations.

6. Compute component-wise multiplication

$$\hat{\mathbf{c}}_k = \hat{\mathbf{a}}_k \hat{\mathbf{b}}_k.$$

This requires $\mathcal{O}(Nn)$ operations.

7. Compute

$$\mathbf{c}_k = \text{IFFT}(\hat{\mathbf{c}}_k).$$

This requires $\mathcal{O}(Nn \log n)$ operations.

Table 2 Data used in the various examples. The polynomial for the final conformal mapping is of the form $\varphi_1(z) = \sum_{k=0}^{n-1} c_k z^k$ with $n = 16$. Coefficients not listed are zero. In all examples, we used $N = 20$ discretisation points in time

Annotation	Coefficients	Choice of α	Type
Example 1 (a,b)	$c_0 = 0$	$\alpha = 0.1$ in (a)	Scaling
	$c_1 = 0.5$	$\alpha = 100$ in (b)	
Example 2 (a,b)	$c_0 = 0$	$\alpha = 0.1$ in (a)	Rotation
	$c_1 = \exp(0.4\pi i)$	$\alpha = 100$ in (b)	
Example 3 (a,b)	$c_0 = 0.0185475$	$\alpha = 0.1$ in (a)	Non-linear
	$c_1 = 0.8034225$	$\alpha = 10$ in (b)	
	$c_2 = -0.13933275$		
	$c_3 = -0.23849625$		
	$c_4 = -0.18597975$		
	$c_5 = -0.0125472$		
	$c_6 = 0.18020775$		
	$c_7 = 0.27937125$		
Example 4 (a,b)	$c_0 = 0.00674 + 0.053125i$	$\alpha = 0.1$ in (a)	Non-linear
	$c_1 = 0.77654 + 0.103125i$	$\alpha = 10$ in (b)	
	$c_2 = 0.109424 + 0.103125i$		
	$c_3 = -0.052777 + 0.103125i$		
	$c_4 = -0.115049 + 0.103125i$		
	$c_5 = -0.0409141 + 0.103125i$		
	$c_6 = 0.126201 + 0.103125i$		
	$c_7 = 0.288402 + 0.103125i$		

8. Let ϕ_k be the polynomial with Taylor coefficients given by c_k . Then compute

$$A_2 = \frac{h}{2} \sum_{k=0}^{N-1} \langle\langle \phi_k, \phi_k \rangle\rangle_{L^2(\mathbb{D})}.$$

This requires $\mathcal{O}(Nn)$ operations.

Finally, the discrete action is now given by $A_d(\varphi_0, \dots, \varphi_N) = A_1 + A_2$.

6 Experimental Results

In this section we use the numerical method developed in the previous section to confirm that the discrete Lagrangian method is able to calculate geodesics (solutions to (4)) with moderately large deformations. The purpose of these

experiments is to demonstrate our numerical method: we leave further investigations of conformal image registration for another paper. In all the examples we compute geodesics starting from the identity $\varphi_0(z) = z$ and ending at some polynomial φ_1 (of relatively low order). We consider both simple linear and heavily non-linear warpings. The simulations are carried out with two different values of the parameter α to illustrate how the geodesics depend on the metric. The data is given in Table 2.

Figure 2 shows the geodesics corresponding to scaling and rotation. Notice that the geodesics stay in the submanifold $\text{Lin}(\mathbb{D}, \mathbb{R}^2)$, as predicted by Proposition 1. Also notice the difference between large and small α . For small α , the scaling coefficient behaves (almost) like $\frac{d^2}{dt^2} c_1^2 = 0$, which is the solution of Eq. (7) with $\alpha = 0$, while the scaling coefficient behaves (almost) like $\frac{d^2}{dt^2} c_1 = 0$, which is the asymptotic solution of Eq. (7) as $\alpha \rightarrow \infty$.

Figure 3 shows the geodesics corresponding to various non-linear transformations. Although the differences in the geodesic paths for different values of α are small, we notice

Fig. 2 In Example 1, geodesics in the H_α^1 metric connecting the identity $z \mapsto z$ to $z \mapsto 0.5z$ are calculated using the discrete Lagrangian method. **a** $\alpha = 0.1$ and **b** $\alpha = 100$. Both geodesics coincide with the analytic solution to Eq. (7). In Example 2, the geodesic connecting $z \mapsto z$ to $z \mapsto e^{0.4\pi i} z$ is calculated, again matching the analytic solution perfectly, illustrating the effect of α

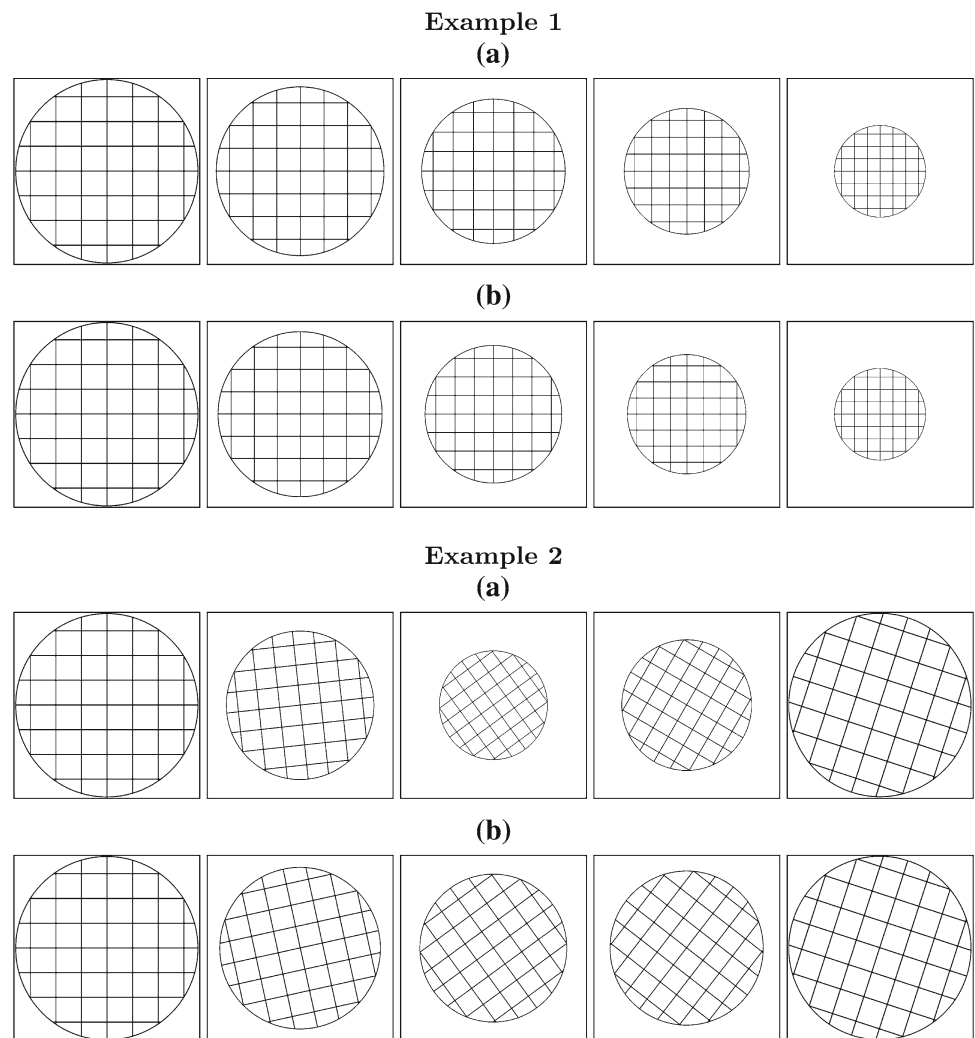
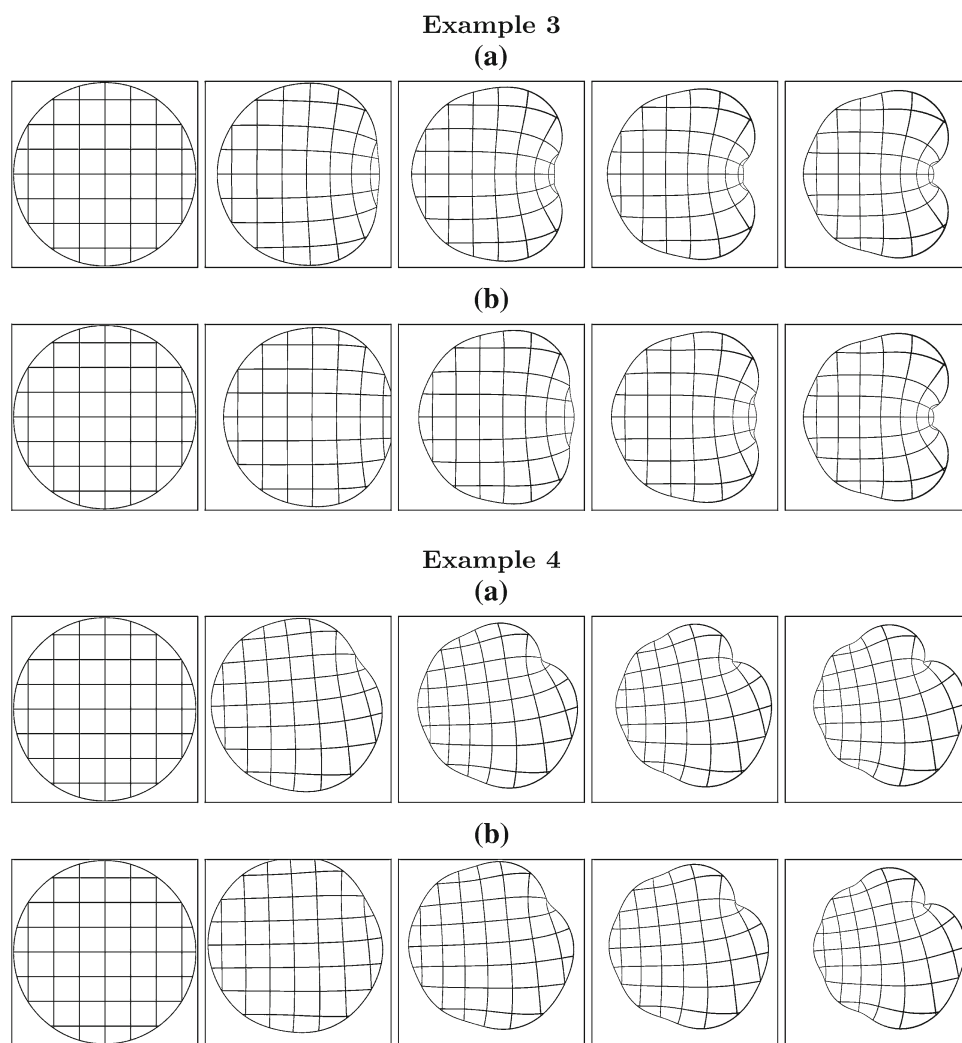


Fig. 3 Examples 3 and 4 illustrate two geodesics in the H_α^1 metric on conformal embeddings calculated using the discrete Lagrangian method. **a** $\alpha = 0.1$ and **b** $\alpha = 100$. In both examples, the target diffeomorphism φ_1 has been chosen to be highly non-linear (see Table 2 for the exact data used). Little difference is visible to the eye between the two values of α : in Example 3, a small bump on the right side of the boundary behaves differently



that for higher values of α the geodesic are “more regular” at the cost of occupying more volume. This is especially clear in Example 3, where, halfway through the geodesic, the “bump” on the right of the shape behaves differently for the two values of α .

We anticipate that the method will allow us to study the metric geometry of the conformal embeddings as was done by [Michor and Mumford \(2006\)](#) for metrics on planar curves and to determine, for example, which conformal embeddings are markedly closer in one metric than another, and how the geodesic paths differ between different metrics and between different groups, for example, by passing to a smaller group (e.g. the Möbius group) or to a larger one (e.g. the full diffeomorphism pseudogroup for embeddings).

7 Conclusions

Motivated by the preference of [Thompson \(1942\)](#) for ‘simple’ warps in his examples of how images of one species can be

deformed into those of a related species, we have derived the geodesic equation for planar conformal diffeomorphisms using the H_α^1 metric. We have chosen conformal warps as they were used by Thompson, and are very simple. Of course, the animals that Thompson was interested in are actually three-dimensional, and for any number of dimensions bigger than 2 the set of conformal warps is rather restricted. However, our intention is to start with conformal warps and to continue to build progressively more complex sets of deformations, rather than working always in the full diffeomorphism group as is conventional.

The conformal warps admit the rigid transformations of rotation and translation as special cases, and we have shown that these linear conformal transformations are totally geodesic in the conformal warps with respect to the H_α^1 metric that we have considered.

We have also provided a numerical discretization of the geodesic equation, and used it to demonstrate the effects of the parameters of the conformal warps. In future work we will use the algorithm that we have developed here to

perform image registration based on conformal warps using the LDDMM framework and apply it to images such as real examples of those drawn by Thompson.

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References

- Arnold, V. I., & Khesin, B. A. (1998). *Topological methods in hydrodynamics. Volume 125 of applied mathematical sciences*. New York: Springer.
- Beg, M. (2003). Variational and computational methods for flows of diffeomorphisms in image matching and growth in computational anatomy. PhD thesis, John Hopkins University.
- Beg, M. F., Miller, M. I., Trouvé, A., & Younes, L. (2005). Computing large deformation metric mappings via geodesic flows of diffeomorphisms. *International Journal of Computer Vision*, *61*, 139–157.
- Bruveris, M., Gay-Balmaz, F., Holm, D. D., & Ratiu, T. S. (2011). The momentum map representation of images. *Journal of Nonlinear Science*, *21*, 115–150.
- Dupuis, P., & Grenander, U. (1998). Variational problems on flows of diffeomorphisms for image matching. *Quarterly of Applied Mathematics*, *LVI*, 587–600.
- Ebin, D. G., & Marsden, J. E. (1970). Groups of diffeomorphisms and the notion of an incompressible fluid. *Annals of Mathematics*, *92*, 102–163.
- Gay-Balmaz, F., Marsden, J., & Ratiu, T. (2012). Reduced variational formulations in free boundary continuum mechanics. *Journal of Nonlinear Science*, *22*, 463–497.
- Hamilton, R. S. (1982). The inverse function theorem of Nash and Moser. *Bulletin of the American Mathematical Society (New Series)*, *7*, 65–222.
- Holm, D. D., & Marsden, J. E. (2005). Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation. In *The breadth of symplectic and Poisson geometry*. Progress in Mathematics (Vol. 232, pp. 203–235). Boston, MA: Birkhäuser.
- Joshi, S., & Miller, M. (2000). Landmark matching via large deformation diffeomorphisms. *IEEE Transactions on Image Processing*, *9*, 1357–1370.
- Khesin, B., & Wendt, R. (2009). *The Geometry of Infinite-dimensional Groups. Volume 51 of a series of modern surveys in mathematics*. Berlin: Springer.
- Lang, S. (1999). *Fundamentals of differential geometry. Volume 191 of Graduate texts in mathematics*. New York: Springer.
- Marsden, J. E., & West, M. (2001). Discrete mechanics and variational integrators. *Acta Numerica*, *10*, 357–514.
- Marsland, S., McLachlan, R.I., Modin, K., & Perlmutter, M. (2011a). On a geodesic equation for planar conformal template matching. In Proceedings of the 3rd MICCAI workshop on mathematical foundations of computational anatomy (MFCA'11), Toronto.
- Marsland, S., McLachlan, R.I., Modin, K., & Perlmutter, M. (2011b). Application of the hodge decomposition to conformal variational problems. arXiv:1203.4464v1 [math.DG].
- Michor, P. W., & Mumford, D. (2006). Riemannian geometries on spaces of plane curves. *Journal of European Mathematical Society (JEMS)*, *8*, 1–48.
- Miller, M. I., & Younes, L. (2001). Group actions, homeomorphisms, and matching: A general framework. *International Journal of Computer Vision*, *41*, 61–84.
- Modin, K., Perlmutter, M., Marsland, S., & McLachlan, R. I. (2011). On Euler–Arnold equations and totally geodesic subgroups. *Journal of Geometry and Physics*, *61*, 1446–1461.
- Sharon, E., & Mumford, D. (2006). 2D-shape analysis using conformal mapping. *International Journal of Computer Vision*, *70*, 55–75.
- Shkoller, S. (1998). Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics. *Journal of Functional Analysis*, *160*, 337–365.
- Thompson, D. (1942). *On growth and form*. New York: Cambridge University Press.
- Trouvé, A. (1995). An infinite dimensional group approach for physics based models in patterns recognition. Technical report, Ecole Normale Supérieure.
- Trouvé, A. (1998). Diffeomorphisms groups and pattern matching in image analysis. *International Journal of Computer Vision*, *28*, 213–221.
- Wallace, A. (2006). D’Arcy Thompson and the theory of transformations. *Nature Reviews Genetics*, *7*, 401–406.
- Younes, L. (2010). *Shapes and diffeomorphisms. Applied mathematical sciences*. New York: Springer.