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FAST TRACK COMMUNICATION

The structure of a set of vector fields on Poisson manifolds

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Abstract
We show that the Lie bracket of an arbitrary vector field with a Hamiltonian vector field is the sum of a Hamiltonian vector field and an energy-preserving vector field, but that not all vector fields can be so decomposed.

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We present an algebraic property of a set of vector fields on a symplectic or Poisson manifold that, while simple, does not appear in the standard sources (e.g. [1, 2]). Its novel feature is that it relates *non*-Hamiltonian and Hamiltonian vector fields. It was discovered in the course of an investigation of series of elementary differentials of a vector field used in geometric numerical integration [3].

Let $(P, \{,\})$ be an n-dimensional Poisson manifold and $H: P \to \mathbb{R}$ a real (C^∞) function on P that we call the energy. Let $\mathfrak{X}$ be the Lie algebra of (C^∞) vector fields on P. The two structures $\{,\}$ and H endow $\mathfrak{X}$ with a distinguished element, namely the Hamiltonian vector field X_H, and with two Lie subalgebras: $\mathfrak{X}_{\rm Ham}$, the Lie algebra of Hamiltonian vector fields on P, and $\mathfrak{X}_H$, the Lie algebra of energy- (i.e. H-) preserving vector fields on P. The Hamiltonian vector field X_H lies in both $\mathfrak{X}_{\rm Ham}$ and $\mathfrak{X}_H$.

Elements of $\mathfrak{X}_H$ are described locally by $n-1$ scalar functions, while elements of $\mathfrak{X}_{\rm Ham}$ are described by single scalar functions. Thus, it makes sense to ask if an arbitrary vector field X (described by n scalar functions) is the sum of a Hamiltonian vector field and an energy-preserving vector field. We shall see that this is (i) true locally near regular points of X_H, (ii) not necessarily true near singular points of X_H and (iii) true globally when $X = [Z, X_H]$ is the Lie bracket of an arbitrary vector field Z with X_H. This provides a universal constraint on the range of ${\rm ad}_{X_H}$. We also have an algebraic description as follows.

Proposition 1. $[\mathfrak{X}, X_H] \subset \mathfrak{X}_{\rm Ham} + \mathfrak{X}_H$.

Proof. Let $Z \in \mathfrak{X}$. We will show that the Hamiltonian part of $[Z, X_H]$ can be taken to be $X_{Z(H)}$. This will be true if the remainder $[Z, X_H] - X_{Z(H)}$ is energy-preserving, which can be checked as follows:

$$\begin{aligned}([Z, X_H] - X_{Z(H)})(H) &= Z(X_H(H)) - X_H(Z(H)) - X_{Z(H)}(H)\\ &= 0 - X_H(Z(H)) - \{Z(H), H\}\\ &= -\{H, Z(H)\} - \{Z(H), H\}\\ &= 0.\end{aligned}$$

□

The decomposition is of course only unique up to elements of $\mathfrak{X}_{\mathrm{Ham}} \cap \mathfrak{X}_H$, the Hamiltonian vector fields that conserve H.

Proposition 2. *Let $H \in C^\infty(P)$ and Z be an arbitrary vector field on P. (i) In the neighborhood of a regular point of X_H, there is a Hamiltonian function K and an energy-preserving vector field Y such that $Z = X_K + Y$. (ii) In the neighborhood of a singular point of X_H, such K and Y need not exist.*

Proof. For (i), we have to solve $Z = X_K + Y, Y(H) = 0$ for K and Y. This requires $Z(H) = X_K(H) + Y(H) = X_K(H) = -X_H(K)$, that is, the derivative of K along X_H is prescribed. In the neighborhood of any regular (i.e. nonzero) point of X_H, there is a local cross-section transverse to X_H. Take K arbitrary on this cross-section and let $Z(H) = -X_H(K)$ determine K uniquely away from the cross-section. Then $Y(H) = Z(H) - X_K(H) = 0$, that is, Y is energy-preserving. For (ii), take $P = T^*\mathbb{R}$ with coordinates (q, p) and the canonical Poisson bracket, and let $H = \frac{1}{2}(q^2 + p^2)$ so that the orbits of X_H are circles centered on the origin. The origin is a singular point of X_H, and $X_H(K) = -Z(H)$ has a solution near $(0, 0)$ for K only if the integral around each circle centered on the origin of $Z(H)$ is zero, but there exist Z (for example, $Z = q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p}$, $Z(H) = q^2 + p^2$) for which these integrals are nonzero. □

If P is symplectic, the restriction to regular points of X_H is equivalent to a restriction to regular points of H. But if P is not symplectic, one can take, for example, H to be a Casimir of the Poisson bracket. Then X_K and Y are both energy-preserving, so only energy-preserving vector fields can be decomposed as a sum $X_K + Y$.

One can also ask if the decomposition results in propositions 1 and 2 extend to subalgebras of $\mathfrak{X}$. For the decomposition to make sense, X_H must be an element of the subalgebra. We study this for four cases: (1) vector fields with a symmetry, (2) volume-preserving vector fields, (3) elementary differentials of a vector field and (4) vector fields with a first integral.

Case 1. If a (discrete or continuous) symmetry acts on P and the Poisson bracket, and if Z and H share the symmetry, then X_H also has the symmetry, and hence so do $Z(H)$, $X_{Z(H)}$ and the remainder (the energy-preserving part of the decomposition) $[Z, X_H] - X_{Z(H)}$. Thus, proposition 1 holds for symmetric vector fields. Proposition 2(i) does not hold in the symmetric case, by the following counterexample. Let $P = T^*\mathbb{R}$ as in proposition 2, let $H = p, X_H = \frac{\partial}{\partial q}$, and let the symmetry be translation in the q direction. The invariant Hamiltonians are functions of p only and their vector fields have $\dot{p} = 0$, i.e. they are energy-preserving. But not all invariant vector fields are energy-preserving. Tracing through the proof of proposition 2(i) in this case shows the problem: if $Z = a(p)\frac{\partial}{\partial q} + b(p)\frac{\partial}{\partial p}$, the differential equation for K, $\frac{\partial K}{\partial q} = -b(p)$, is invariant, but its solution $K = -qb(p) + c(p)$ is not invariant (and nor is its Hamiltonian vector field $X_K = (-qb'(p) + c'(p))\frac{\partial}{\partial q} + b(p)\frac{\partial}{\partial p}$). An invariant differential equation need not have any invariant solutions. Proposition 2(ii) holds in the symmetric case, but is vacuous (take the trivial group).

Case 2. If Z and all Hamiltonian vector fields are volume-preserving, then so are $[Z, X_H]$ and $X_{Z(H)}$; so proposition 1 holds. Propositions 2(i) and (ii) hold too, because the volume-preserving nature of Z does not enter the argument.

Case 3. On $P = \mathbb{R}^n$ with a constant Poisson structure, $X_H =: f$, the linear combinations of f and its derivatives (the *elementary differentials* of f) span a Lie algebra $\mathfrak{B} := \mathrm{span}(f, f'(f), f''(f, f), f'(f'(f)), \ldots)$. It is typically countably infinite dimensional. Some of its elements are energy-preserving (e.g. $f'(f'(f))$) and some are Hamiltonian (e.g. $f''(f, f) - 2f'(f'(f))$). If $Z \in \mathfrak{B}$ then $[Z, X_H] \in \mathfrak{B}$ and $X_{Z(H)} \in \mathfrak{B}$, so the decomposition holds in elementary differentials, too. Proposition 2(ii) holds too; the Hamiltonian and energy-preserving elementary differentials, and those not in their span, can be enumerated [3]. Proposition 2(i) does not hold in this case, because a local decomposition would determine the elementary differentials in a global decomposition.

Case 4. For vector fields with a given first integral, the decomposition result does not hold even locally. Consider $P = T^*\mathbb{R}^2$ with the canonical bracket and vector fields with first integral q_1. Then $\frac{\partial H}{\partial p_1} = 0$ and the q_1-component of Z is zero and we seek a decomposition $Z = X_K + Y$ with $Y(H) = 0$ and $\frac{\partial K}{\partial p_1} = 0$. As in the proof of proposition 2, we need $X_H(K) = -Z(H) = -Z_{q_2}(q_2, p_1, p_2)\frac{\partial H}{\partial q_2} - Z_{p_2}(q_2, p_1, p_2)\frac{\partial H}{\partial p_2}$; clearly, K cannot be independent of p_1 for all such Z. Replacing Z by, say, $[\widetilde{Z}, X_H]$ does not help, so proposition 1 does not hold in this case either.

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