IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **47** (2014) 145204 (20pp) doi:10.1088/1751-8113/47/14/145204

On conformal variational problems and free boundary continua

Stephen Marsland[1], **Robert I McLachlan**[2], **Klas Modin**[3] **and Matthew Perlmutter**[4]

[1] School of Engineering and Advanced Technology, Massey University, Private Bag 11 222, Palmerston North 4442, New Zealand
[2] Institute of Fundamental Sciences, Massey University, Private Bag 11 222, Palmerston North 4442, New Zealand
[3] Mathematical Sciences, Chalmers University of Technology, Gothenburg, Sweden
[4] Departamento de Matemática, Universidade Federal de Minas Gerais, Belo Horizonte, Brazil

E-mail: klas.modin@chalmers.se

Received 3 September 2013, revised 5 March 2014
Accepted for publication 5 March 2014
Published 26 March 2014

Abstract

We develop a framework for deriving governing partial differential equations for variational problems on spaces of conformal mappings. The main motivation is to obtain differential equations for the conformal motion of free boundary continua, of interest in image and shape registration. A fundamental tool in the paper, the Hodge–Morrey–Friedrichs decompositions of differential forms on manifolds with boundaries, is used to identify the orthogonal complement of the subspace of conformal mappings. A detailed presentation of these decompositions is included in the paper.

Keywords: calculus of variations, conformal embeddings, free boundary continua, Hodge decomposition of vector fields and differential forms
PACS numbers: 02.30.Xx, 02.40.Tt, 02.40.Yy, 46.15.Cc
Mathematics Subject Classification: 30C70, 58A14, 35A15, 58E30

1. Introduction

It is well known that the Euler equation of fluid dynamics is a geodesic equation on the group of volume preserving diffeomorphisms expressed as an Euler–Poincaré equation on the Lie algebra of volume preserving vector fields [1, 2]. On domains of $\mathbb{R}^2$ or $\mathbb{R}^3$, the derivation of the strong form of the Euler equations starts from a variational principle and relies on the Helmholtz decomposition in order to identify the L^2 orthogonal complement of the space of volume preserving vector fields inside the space of all vector fields. A generalization to arbitrary compact Riemannian manifolds, with or without boundary, is obtained by identifying vector fields with 1-forms (by contraction with the metric), and then using the Hodge decomposition for manifolds with boundary in order to identify the L^2 orthogonal complement [3].

In this paper we are interested in techniques for deriving governing partial differential equations for variational problems on spaces of conformal mappings. Motivated by conformal image and shape registration, the authors derived in [4] the weak form of a geodesic equation on the space of planar conformal embeddings. One may regard this equation as the conformal analogue of the 'EPDiff' equation used in certain fluid models [5] and in the LDDMM method in image registration [6]. The motivation for studying conformal shape registration come from the suggestion [7] that changes of shape that are caused by growth—whether on an organism or evolutionary timescale—can be conformal.

Our work originated from the search for a strong form of the weak geodesic equation in [4]; we realized that a framework for deriving governing equations for conformal variational problems is missing in the literature. Our primary purpose here is to provide such a framework. In addition, we realized that the main tool for deriving the Euler equations, the *Hodge–Morrey–Friedrich decompositions*, can also be used for conformal variational problems, by modifying the isomorphism between vector fields and 1–forms. A secondary purpose of the paper is to give a general presentation of how various Lie subalgebras of vector fields can be identified with one or several components in the Hodge–Morrey–Friedrich decompositions, thus identifying the L^2 orthogonal complement necessary for deriving governing equations.

The paper is organized as follows. In section 2 we give a detailed review of three types of vector field subalgebras: volume preserving, symplectic, and conformal. We work in the category of Fréchet–Lie algebras, and we give proofs that the subalgebras considered are proper Fréchet–Lie subalgebras. In this section it is not required that the underlying manifold is compact.

In section 3 we review the Hodge–Morrey–Friedrichs decompositions for compact manifolds with boundary. We state a complete Hodge decomposition, involving six spaces, and we show that these spaces can be characterized in terms of the kernel and image spaces of the differential and codifferential. Although our results are simple consequences of the Hodge–Morrey–Friedrichs decompositions, we have not found them elsewhere in the literature. The decomposition of differential forms into six orthogonal subspaces is the finest decomposition obtained by applying the operations of exterior derivative, codifferential, set intersection, and orthogonal complement to differential forms. The special case of 2-manifolds is studied in further detail, and the unit disc, the standard annulus, the torus, and the sphere are given as examples.

In section 4 we show how to use the Hodge decomposition to obtain L^2 orthogonal decompositions of vector fields. Altogether, we derive nine different decompositions, which are summarized in table 1. Proposition 4.1 gives a decomposition of vector fields on a flat 2-manifold, involving conformal vector fields as one of the components. The section ends with various examples and one counter-example.

In section 5, results from the previous sections are used to accomplish the main objective of determining the governing differential equations for three variational problems on the space of conformal vector fields on a simply connected bounded domain of $\mathbb{R}^2$. This involves two techniques that do not arise in the standard calculus of variations: the projection to conformal vector fields (proposition 5.2) and a result on integration by parts in the conformal setting without boundary terms (proposition 5.4). Together with the decomposition in proposition 4.1 these techniques allow, in subsection 5.5, a complete derivation of the strong form of the equations of motion for the free boundary problem of geodesic motion on the space of conformal embeddings studied in [4].

Table 1. Various L^2 orthogonal decompositions of vector fields based on the Hodge decomposition. The first column specifies the required setting. M is a manifold, possibly with boundary, and the upper index denotes its dimension. The second column specifies the isomorphism used to identify vector fields with 1-forms. The third column gives various L^2 orthogonal decompositions of $\mathfrak{X}(M)$, and corresponding components in the Hodge decomposition (via the contraction map). The first component in each decomposition is a Lie subalgebra of vector fields.

Setting	Isomorphism	Decompositions
M^n, g	$\xi \mapsto \mathrm{i}_\xi \mathsf{g}$	$\mathfrak{X}(M) = \underbrace{\mathfrak{X}_{\mathrm{vol}}(M)}_{\bigoplus_{l\neq 1} A^1_l(M)^\sharp} \oplus \underbrace{\mathrm{grad}(\mathcal{F}_0(M))}_{A^1_1(M)^\sharp}$
		$\mathfrak{X}(M) = \underbrace{\mathfrak{X}_{\mathrm{vol,t}}(M)}_{\bigoplus_{l\in\{2,3,5\}} A^1_l(M)^\sharp} \oplus \underbrace{\mathrm{grad}(\mathcal{F}(M))}_{\bigoplus_{l\in\{1,4,6\}} A^1_l(M)^\sharp}$
		$\mathfrak{X}(M) = \underbrace{\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)}_{\bigoplus_{l\in\{2,5,6\}} A^1_l(M)^\sharp} \oplus \underbrace{\mathrm{grad}(\mathcal{F}_0(M))}_{A^1_1(M)^\sharp} \oplus \underbrace{\mathcal{H}^1_{\mathrm{n}}(M)^\sharp}_{\bigoplus_{l\in\{3,4\}} A^1_l(M)^\sharp}$
		$\mathfrak{X}(M) = \underbrace{\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol,t}}(M)}_{\bigoplus_{l\in\{2,5\}} A^1_l(M)^\sharp} \oplus \underbrace{\mathrm{grad}(\mathcal{F}(M))}_{\bigoplus_{l\in\{1,4,6\}} A^1_l(M)^\sharp} \oplus \underbrace{\mathcal{H}^1_0(M)^\sharp}_{A^1_3(M)^\sharp}$
M^{2n}, g, ω (almost Kähler)	$\xi \mapsto \mathrm{i}_\xi \omega$	$\mathfrak{X}(M) = \underbrace{\mathfrak{X}_{\omega}(M)}_{\bigoplus_{l\neq 2} A^1_l(M)^\sharp} \oplus \underbrace{\delta\Omega^2_{\mathrm{t}}(M)^\sharp}_{A^1_2(M)^\sharp}$
		$\mathfrak{X}(M) = \underbrace{\mathfrak{X}_{\omega,\mathrm{t}}(M)}_{\bigoplus_{l\in\{1,3,4\}} A^1_l(M)^\sharp} \oplus \underbrace{\delta\Omega^2(M)^\sharp}_{\bigoplus_{l\in\{2,5,6\}} A^1_l(M)^\sharp}$
		$\mathfrak{X}(M) = \underbrace{\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)}_{\bigoplus_{l\in\{1,4,6\}} A^1_l(M)^\sharp} \oplus \underbrace{\delta\Omega^2_{\mathrm{t}}(M)^\sharp}_{A^1_2(M)^\sharp} \oplus \underbrace{\mathcal{H}^1_{\mathrm{t}}(M)^\sharp}_{\bigoplus_{l\in\{3,5\}} A^1_l(M)^\sharp}$
		$\mathfrak{X}(M) = \underbrace{\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol,t}}(M)}_{\bigoplus_{l\in\{1,4\}} A^1_l(M)^\sharp} \oplus \underbrace{\delta\Omega^2(M)^\sharp}_{\bigoplus_{l\in\{2,5,6\}} A^1_l(M)^\sharp} \oplus \underbrace{\mathcal{H}^1_0(M)^\sharp}_{A^1_3(M)^\sharp}$
M^2, g (flat)	$\xi \mapsto \mathrm{i}_\xi \mathsf{g}$	$\mathfrak{X}(M) = \underbrace{\mathfrak{X}_{\mathrm{con}}(M)}_{\bigoplus_{l\neq\{1,2\}} A^1_l(M)^\sharp} \oplus \underbrace{\overline{\mathrm{grad}}(\mathcal{F}_0(M))}_{A^1_1(M)^\sharp} \oplus \underbrace{\overline{\mathrm{sgrad}}(\mathcal{F}_0(M))}_{A^1_2(M)^\sharp}$

2. Lie algebras of vector fields

In this section, let M be an n-manifold, possibly with boundary, such that M is either compact, or can be equipped with a countable sequence of compact sets $K_i \subset M$ such that each compact subset $U \subset M$ is contained in one of K_i. In this case, the linear space $\mathcal{F}(M)$ of smooth real valued function on M can be equipped with a sequence of semi-norms, making it a Fréchet space (see e.g. [8] for details). In turn, this induces a Fréchet topology on the space $\mathcal{T}_r^s(M)$ of smooth tensor fields on M of finite order. In particular, the space $\mathfrak{X}(M)$ of vector fields, and the space $\Omega^k(M)$ of k-forms are Fréchet spaces. Furthermore, the topologies are 'compatible' with each other, in the sense that any partial differential operator with smooth coefficients between any two spaces of tensor fields is a smooth map [8, section II.2.2]. In particular, the Lie derivative map

$$\mathfrak{X}(M) \times \mathcal{T}_r^s(M) \ni (\xi, t) \mapsto \pounds_\xi t \in \mathcal{T}_r^s(M)$$

is smooth, which in turn implies that $\mathfrak{X}(M)$ is a Fréchet–Lie algebra with Lie bracket given by $[\xi, \eta] = -\pounds_\xi \eta$ (this bracket fulfils the Jacobi identity).

Recall that a subspace of a Fréchet–Lie algebra is called a Fréchet–Lie subalgebra if it is topologically closed, and also closed under the Lie bracket. In the case when M has a boundary, it holds that the subspace $\mathfrak{X}_{\mathrm{t}}(M)$ of vector fields that are tangential to the boundary is a Fréchet–Lie subalgebra. Basically, this is the only subalgebra which can be obtained intrinsically, without introducing any further structures on M. In the remainder of this section we review some other well known subalgebras of vector fields which require extra structure on the manifold.

2.1. Volume preserving vector fields

Assume that M is orientable, and let M be equipped with a volume form, denoted vol. The set of volume preserving vector fields is then given by $\mathfrak{X}_{\mathrm{vol}}(M) = \{\xi \in \mathfrak{X}(M); \pounds_\xi \mathrm{vol} = 0\}$. It is clear that this is a linear subspace of $\mathfrak{X}(M)$. Recall that the divergence with respect to vol is the partial differential operator $\mathrm{div} : \mathfrak{X}(M) \to \mathcal{F}(M)$ defined by $\pounds_\xi \mathrm{vol} = \mathrm{div}(\xi)\mathrm{vol}$ for all $\xi \in \mathfrak{X}(M)$. Thus, since the volume form is strictly non-zero, it holds that $\xi \in \mathfrak{X}_{\mathrm{vol}}(M)$ if and only if $\mathrm{div}(\xi) = 0$.

Proposition 2.1. *$\mathfrak{X}_{\mathrm{vol}}(M)$ is a Fréchet–Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. The differential operator $\mathrm{div} : \mathfrak{X}(M) \to \mathcal{F}(M)$ is smooth in the Fréchet topology. In particular, it is continuous, so the preimage of the closed set $\{0\} \in \mathcal{F}(M)$, which is equal to $\mathfrak{X}_{\mathrm{vol}}(M)$, is also closed. Thus, $\mathfrak{X}_{\mathrm{vol}}(M)$ is a topologically closed subspace of $\mathfrak{X}(M)$.

Next, let $\xi, \eta \in \mathfrak{X}_{\mathrm{vol}}(M)$. Then

$$\pounds_{[\xi,\eta]}\mathrm{vol} = \pounds_{\pounds_\eta \xi}\mathrm{vol} = \pounds_\eta \underbrace{\pounds_\xi \mathrm{vol}}_{0} - \pounds_\xi \underbrace{\pounds_\eta \mathrm{vol}}_{0} = 0.$$

Thus, $\mathfrak{X}_{\mathrm{vol}}(M)$ is closed under the Lie bracket, which finishes the proof. □

In the case when M has a boundary, it also holds that the subspace $\mathfrak{X}_{\mathrm{vol,t}}(M) := \mathfrak{X}_{\mathrm{vol}}(M) \cap \mathfrak{X}_{\mathrm{t}}(M)$ is a Fréchet–Lie subalgebra. This follows immediately since both $\mathfrak{X}_{\mathrm{vol}}(M)$ and $\mathfrak{X}_{\mathrm{t}}(M)$ are Fréchet–Lie subalgebras.

Next, consider the subspace of *exact* divergence free vector fields given by

$$\mathfrak{X}_{\mathrm{vol}}^{\mathrm{ex}}(M) = \{\xi \in \mathfrak{X}(M); \mathrm{i}_\xi \mathrm{vol} \in \mathrm{d}\Omega^{n-2}(M)\}.$$

The following result is well known (see e.g. [2]).

Proposition 2.2. *$\mathfrak{X}_{\mathrm{vol}}^{\mathrm{ex}}(M)$ is a Fréchet–Lie subalgebra of $\mathfrak{X}(M)$ and an ideal in $\mathfrak{X}_{\mathrm{vol}}(M)$.*

Proof. Topological closedness follows since the maps $\Omega^{n-1}(M) \ni \mathrm{i}_\xi \mathrm{vol} \mapsto \xi \in \mathfrak{X}(M)$ and $\mathrm{d} : \Omega^{n-2}(M) \to \Omega^{n-1}(M)$ are smooth in the Fréchet topology. Next, if $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ then ξ is divergence free since $\pounds_\xi \mathrm{vol} = \mathrm{d}\mathrm{i}_\xi \mathrm{vol} = 0$. Finally, if $\eta \in \mathfrak{X}_{\mathrm{vol}}(M)$ then $\mathrm{i}_{\pounds_\eta \xi} \mathrm{vol} = \pounds_\eta \mathrm{i}_\xi \mathrm{vol} = \pounds_\eta \,\mathrm{d}\alpha = \mathrm{d}\pounds_\eta \alpha \in \mathrm{d}\Omega^{n-2}(M)$, so $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ is an ideal in $\mathfrak{X}_{\mathrm{vol}}(M)$. □

Continuing as before, we also obtain the smaller Fréchet–Lie subalgebra of tangential exact divergence free vector fields, by

$$\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol,t}}(M) = \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M) \cap \mathfrak{X}_{\mathrm{t}}(M).$$

The space of volume preserving vector fields is of importance in fluid mechanics. In particular, the motion of an incompressible ideal fluid is described by a differential equation evolving on the phase space $\mathfrak{X}_{\mathrm{vol,t}}(M)$, which is the Lie algebra of the set of volume preserving diffeomorphisms of M [1].

2.2. Symplectic vector fields

Let M be equipped with a symplectic structure, i.e., a closed non-degenerate 2-form ω. Then the subspace of symplectic vector fields on M is given by $\mathfrak{X}_\omega(M) = \{\xi \in \mathfrak{X}(M);\, \pounds_\xi \omega = 0\}$.

Proposition 2.3. *$\mathfrak{X}_\omega(M)$ is a Fréchet–Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. The map $\mathfrak{X}(M) \ni \xi \mapsto \pounds_\xi \omega \in \Omega^2(M)$ is smooth, so its preimage of $\{0\} \in \Omega^2(M)$ is topologically closed. Thus, $\mathfrak{X}_\omega(M)$ is topologically closed in $\mathfrak{X}(M)$. Further, if $\xi, \eta \in \mathfrak{X}_\omega(M)$, then

$$\pounds_{[\xi,\eta]}\omega = \pounds_\eta \underbrace{\pounds_\xi \omega}_{0} - \pounds_\xi \underbrace{\pounds_\eta \omega}_{0} = 0.$$

Thus, $\mathfrak{X}_\omega(M)$ is closed under bracket, which concludes the proof. □

The space of Hamiltonian vector fields are those that have a globally defined Hamiltonian. That is,

$$\mathfrak{X}_{\mathrm{Ham}}(M) = \{\xi \in \mathfrak{X}(M);\, \mathrm{i}_\xi \omega \in \mathrm{d}\Omega^0(M)\}.$$

With the same proof as for proposition 2.2, but replacing vol with ω, we get the following result.

Proposition 2.4. *$\mathfrak{X}_{\mathrm{Ham}}(M)$ is a Fréchet–Lie subalgebra of $\mathfrak{X}(M)$ and an ideal in $\mathfrak{X}_\omega(M)$.*

Just as in the volume preserving case, we also have the smaller Fréchet–Lie subalgebras of symplectic and Hamiltonian tangential vector fields,

$$\mathfrak{X}_{\omega,\mathrm{t}}(M) = \mathfrak{X}_\omega(M) \cap \mathfrak{X}_{\mathrm{t}}(M) \quad \text{and} \quad \mathfrak{X}_{\mathrm{Ham,t}}(M) = \mathfrak{X}_{\mathrm{Ham}}(M) \cap \mathfrak{X}_{\mathrm{t}}(M).$$

2.3. Conformal vector fields

Let M be equipped with a Riemannian metric g. Then the subspace of conformal vector fields is given by

$$\mathfrak{X}_{\mathrm{con}}(M) = \{\xi \in \mathfrak{X}(M);\, \pounds_\xi \mathsf{g} = F\mathsf{g},\, F \in \mathcal{F}(M)\}.$$

Thus, if $\xi \in \mathfrak{X}_{\mathrm{con}}(M)$ then ξ preserves the metric up to scaling by a function. In turn, this implies that the infinitesimal transformation generated by ξ preserve angles. Indeed, if $\eta, \psi \in \mathfrak{X}(M)$ are everywhere orthogonal, i.e., $\mathrm{i}_\eta \mathrm{i}_\psi \mathsf{g} = \mathsf{g}(\eta, \psi) = 0$, then $\mathsf{g}(\pounds_\xi \eta, \psi) + \mathsf{g}(\eta, \pounds_\xi \psi) = 0$, which follows since

$$0 = \pounds_\xi(\mathsf{g}(\eta,\psi)) = F\mathsf{g}(\eta,\psi) + \mathsf{g}(\pounds_\xi \eta, \psi) + \mathsf{g}(\eta, \pounds_\xi \psi) = \mathsf{g}(\pounds_\xi \eta, \psi) + \mathsf{g}(\eta, \pounds_\xi \psi).$$

Proposition 2.5. *$\mathfrak{X}_{\mathrm{con}}(M)$ is a Fréchet–Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. We need to show that $\mathfrak{X}_{\mathrm{con}}(M)$ is closed under the Lie bracket and that $\mathfrak{X}_{\mathrm{con}}(M)$ is topologically closed in $\mathfrak{X}(M)$. Let $\xi, \eta \in \mathfrak{X}_{\mathrm{con}}(M)$. Then

$$\pounds_{\pounds_\xi \eta} \mathrm{g} = \pounds_\xi \pounds_\eta \mathrm{g} - \pounds_\eta \pounds_\xi \mathrm{g} = \pounds_\xi (G\mathrm{g}) - \pounds_\eta (F\mathrm{g}) = (\pounds_\xi G - \pounds_\eta F)\mathrm{g},$$

which proves that $[\xi, \eta] = -\pounds_\xi \eta \in \mathfrak{X}_{\mathrm{con}}(M)$.

To prove that $\mathfrak{X}_{\mathrm{con}}(M)$ is topologically closed in $\mathfrak{X}(M)$, we define a map $\Phi : \mathfrak{X}(M) \to \mathcal{T}_2^0(M)$ by $\xi \mapsto \pounds_\xi \mathrm{g}$. This is a smooth map in the Fréchet topology. We notice that $\mathfrak{X}_{\mathrm{con}}(M) = \Phi^{-1}(\mathcal{F}(M)\mathrm{g})$. Since $\mathcal{F}(M)\mathrm{g}$ is topologically closed in $\mathcal{T}_2^0(M)$ it follows from continuity of Φ that its preimage, i.e., $\mathfrak{X}_{\mathrm{con}}(M)$, is topologically closed in $\mathfrak{X}(M)$. □

Notice that the condition $\pounds_\xi \mathrm{g} = F\mathrm{g}$ for a vector field ξ to be conformal is not as 'straightforward' as the conditions for being volume preserving or symplectic, since the function F depends implicitly on ξ. We now work out an explicit coordinate version of this condition in the case when the manifold M is conformally flat.

First, recall that a Riemannian manifold is locally conformally flat if for every element $z \in M$ there exists a neighbourhood U of z and a function $f \in \mathcal{F}(U)$ such that $\mathrm{e}^{2f}\mathrm{g}$ is a flat metric on U. Thus, we may chose local coordinates mapping U conformally to flat Euclidean space, i.e., such that $\mathrm{g} = c \sum_i \mathrm{d}x^i \otimes \mathrm{d}x^i$, with $c = \mathrm{e}^{2f}$. Next, consider a vector field expressed in these coordinates $\xi = \sum_i u^i \frac{\partial}{\partial x^i}$. Then

$$\begin{aligned}
\pounds_\xi \mathrm{g} &= (\pounds_\xi c) \sum_i \mathrm{d}x^i \otimes \mathrm{d}x^i + c \sum_i (\mathrm{d}u^i \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathrm{d}u^i) \\
&= \sum_i \left(2c \frac{\partial u^i}{\partial x^i} + \mathrm{i}_\xi \, \mathrm{d}c \right) \mathrm{d}x^i \otimes \mathrm{d}x^i + \sum_{i<j} c \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) (\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i). \\
&= \sum_i 2 \left(\frac{\partial u^i}{\partial x^i} + \mathrm{i}_\xi \, \mathrm{d}f \right) c \, \mathrm{d}x^i \otimes \mathrm{d}x^i + \sum_{i<j} c \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) (\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i).
\end{aligned}$$

From this we get that $\pounds_\xi \mathrm{g}$ is point-wise parallel with g if the components of ξ fulfil the following $n(n+2)/2 - 1$ relations

$$\begin{cases} \dfrac{\partial u^i}{\partial x^i} - \dfrac{\partial u^{i+1}}{\partial x^{i+1}} = 0 & \forall\, i < n \\ \dfrac{\partial u^i}{\partial x^j} + \dfrac{\partial u^j}{\partial x^i} = 0 & \forall\, i < j. \end{cases} \tag{1}$$

Notice that these equations are independent of the function c. Also, notice that if M is a 2-manifold, these are the Cauchy–Riemann equations.

The coordinate formula (in conformally flat coordinates) for the divergence with respect to the volume form induced by g is given by

$$\mathrm{div}(\xi) = \mathrm{i}_\xi \, \mathrm{d}c + c \sum_i \frac{\partial u^i}{\partial x^i}.$$

Thus, we see that if M is a 2-manifold, and $\xi \in \mathfrak{X}_{\mathrm{con}}(M)$, then locally we have $\pounds_\xi \mathrm{g} = \mathrm{div}(\xi)/c \, \mathrm{g}$. In particular, if (M, g) is a flat 2-manifold, then it holds that $\pounds_\xi \mathrm{g} = \mathrm{div}(\xi)\mathrm{g}$.

3. Hodge decomposition

In this section, let (M, g) be a compact oriented n-dimensional Riemannian manifold possibly with boundary, and let $\Omega^k(M)$ denote the space of smooth k-forms on M. We sometimes use the notation $\mathcal{F}(M)$ for $\Omega^0(M)$. Recall the Hodge star operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$, which is defined in terms of the metric (see [9, chapter 6]). Using the Hodge star, the space $\Omega^k(M)$ is equipped with the L^2 inner product:

$$\langle \alpha, \beta \rangle_M := \int_M \alpha \wedge \star\beta.$$

Up to a boundary integral term, the *codifferential* $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is the formal adjoint of the differential with respect to the L^2 inner product. Indeed, it holds that

$$\langle \mathrm{d}\gamma, \beta \rangle_M = \langle \gamma, \delta\beta \rangle_M + \int_{\partial M} \gamma \wedge \star\beta.$$

The explicit formula is $\delta = (-1)^{(n-k+1)k} \star \mathrm{d} \star$.

The subspaces $\Omega^k_{\mathsf{t}}(M), \Omega^k_{\mathsf{n}}(M) \subset \Omega^k(M)$ of tangential and normal k-forms are defined by

$$\begin{aligned} \Omega^k_{\mathsf{t}}(M) &= \{\alpha \in \Omega^k(M); i^*(\star\alpha) = 0\} \\ \Omega^k_{\mathsf{n}}(M) &= \{\alpha \in \Omega^k(M); i^*(\alpha) = 0\} \end{aligned}$$

where $i : \partial M \to M$ is the inclusion. We also have the subspace of k-forms that vanish on the boundary: $\Omega^k_0(M) = \Omega^k_{\mathsf{t}}(M) \cap \Omega^k_{\mathsf{n}}(M)$. Notice that $\Omega^0_{\mathsf{n}}(M) = \Omega^0_0(M)$, $\Omega^0_{\mathsf{t}}(M) = \Omega^0(M)$, $\Omega^n_{\mathsf{t}}(M) = \Omega^n_0(M)$, and $\Omega^n_{\mathsf{n}}(M) = \Omega^n(M)$. It also holds that $\star\, \Omega^k_{\mathsf{t}}(M) = \Omega^{n-k}_{\mathsf{n}}(M)$.

The following fundamental result is known as the Hodge–Morrey decomposition theorem for manifolds with boundary (see e.g. [10]):

Theorem 3.1. *$\Omega^k(M)$ admits the L^2 orthogonal decomposition*

$$\Omega^k(M) = \mathrm{d}\Omega^{k-1}_{\mathsf{n}}(M) \oplus \delta\Omega^{k+1}_{\mathsf{t}}(M) \oplus \mathcal{H}^k(M),$$

where $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M); \mathrm{d}\alpha = 0, \delta\alpha = 0\}$ are the harmonic k-fields.

The harmonic fields $\mathcal{H}^k(M)$ can be further decomposed in two different ways.

Theorem 3.2. *$\mathcal{H}^k(M)$ admits the L^2 orthogonal decompositions*

$$\begin{aligned} \mathcal{H}^k(M) &= \mathcal{H}^k_{\mathsf{t}}(M) \oplus \{\alpha \in \mathcal{H}^k(M); \alpha = \mathrm{d}\epsilon\} \\ \mathcal{H}^k(M) &= \mathcal{H}^k_{\mathsf{n}}(M) \oplus \{\alpha \in \mathcal{H}^k(M); \alpha = \delta\gamma\} \end{aligned}$$

where $\mathcal{H}^k_{\mathsf{t}}(M)$ and $\mathcal{H}^k_{\mathsf{n}}(M)$ are the harmonic k-fields that are respectively tangential and normal.

Thus, every harmonic field is decomposed into either: (i) a tangential harmonic field plus an exact harmonic field or (ii) a normal harmonic field plus a co-exact harmonic field. Together, theorems 3.1 and 3.2 comprise the Hodge–Morrey–Friedrichs decompositions (see e.g. [10]).

Notice that by combining the first Friedrichs decomposition in theorem 3.2 with the Hodge–Morrey decomposition in theorem 3.1, we obtain an L^2 orthogonal decomposition of exact k-forms as

$$\mathrm{d}\Omega^{k-1}(M) = \mathrm{d}\Omega^{k-1}_{\mathsf{n}}(M) \oplus \{\alpha \in \mathcal{H}^k(M); \alpha = \mathrm{d}\epsilon\}.$$

Indeed, for any $\epsilon \in \Omega^{k-1}(M)$ it cannot hold that $\mathrm{d}\epsilon$ belongs to $\mathcal{H}^k_{\mathsf{t}}(M)$ or $\delta\Omega^{k+1}_{\mathsf{t}}(M)$ since that would imply that $\mathrm{d}\epsilon$ also belongs to $\{\alpha \in \mathcal{H}^k(M); \alpha = \mathrm{d}\epsilon\}$, which is disjoint to both. Similarly, we get

$$\delta\Omega^{k+1}(M) = \delta\Omega^{k+1}_{\mathsf{t}}(M) \oplus \{\alpha \in \mathcal{H}^k(M); \alpha = \delta\gamma\}.$$

Closely related to the harmonic k-fields $\mathcal{H}^k(M)$ are the *harmonic k-forms*, given by $H^k(M) = \{\alpha \in \Omega^k(M); \Delta\alpha = 0\}$, where $\Delta := \delta \circ \mathrm{d} + \mathrm{d} \circ \delta$ is the Laplace–de Rham operator. For a closed manifold it holds that $\mathcal{H}^k(M) = H^k(M)$. However, in the presence of a boundary, $\mathcal{H}^k(M)$ is strictly smaller than $H^k(M)$.

By combining the two versions of the Friedrichs decomposition in theorem 3.2 (taking pairwise intersections of the subspaces) we obtain four mutually orthogonal subspaces of $\mathcal{H}^k(M)$. Altogether, we thus have six mutually orthogonal subspaces of $\Omega^k(M)$, which are given in the following table.

Short name	Definition
$A_1^k(M)$	$\mathrm{d}\Omega_{\mathrm{n}}^{k-1}(M)$
$A_2^k(M)$	$\delta\Omega_{\mathrm{t}}^{k+1}(M)$
$A_3^k(M)$	$\mathcal{H}_{\mathrm{n}}^k(M) \cap \mathcal{H}_{\mathrm{t}}^k(M)$
$A_4^k(M)$	$\mathcal{H}_{\mathrm{n}}^k(M) \cap \mathrm{d}\Omega^{k-1}(M)$
$A_5^k(M)$	$\mathcal{H}_{\mathrm{t}}^k(M) \cap \delta\Omega^{k+1}(M)$
$A_6^k(M)$	$\mathrm{d}\Omega^{k-1}(M) \cap \delta\Omega^{k+1}(M)$

Theorem 3.1 together with theorem 3.2 now yields the following result, which we refer to as the complete Hodge decomposition for manifolds with boundary.

Corollary 3.1. $\Omega^k(M)$ *admits the* L^2 *orthogonal decomposition*

$$\Omega^k(M) = \bigoplus_{l=1}^{6} A_l^k(M).$$

Remark 3.1. If M does not have a boundary, then $A_4^k(M), A_5^k(M), A_6^k(M)$ are trivial, and corollary 3.1 reduces to the ordinary Hodge–Morrey decomposition for closed manifolds. In contrast, if M is the closure of a bounded open subset of $\mathbb{R}^n$ with smooth boundary, then $A_3^k(M)$ is trivial, since a harmonic field on such a manifold which is zero on the boundary, must be zero also in the interior. (The Laplace equation with Dirichlet boundary conditions is well-posed.)

3.1. Characterization in terms of four fundamental subspaces

The differential and codifferential induces four fundamental subspaces of $\Omega^k(M)$ given by

$$\begin{aligned}
&\operatorname{im}\mathrm{d} = \mathrm{d}\Omega^{k-1}(M),\\
&\ker\mathrm{d} = \{\alpha \in \Omega^k(M);\, \mathrm{d}\alpha = 0\},\\
&\operatorname{im}\delta = \delta\Omega^{k+1}(M),\\
&\ker\delta = \{\alpha \in \Omega^k(M);\, \delta\alpha = 0\}.
\end{aligned}$$

Notice that $\operatorname{im}\mathrm{d} \subset \ker\mathrm{d}$ and $\operatorname{im}\delta \subset \ker\delta$. Also notice that the intersection between any two of these subspaces in general is non-empty.

Each of the mutually orthogonal spaces $A_l^k(M)$ can be characterized by orthogonal complements and intersections of the four fundamental subspaces.

Proposition 3.1. *It holds that*

$$\begin{aligned}
A_1^k(M) &= (\ker\delta)^{\perp}\\
A_2^k(M) &= (\ker\mathrm{d})^{\perp}\\
A_3^k(M) &= (\operatorname{im}\mathrm{d})^{\perp} \cap (\operatorname{im}\delta)^{\perp}\\
A_4^k(M) &= \ker\delta \cap \operatorname{im}\mathrm{d} \cap (\operatorname{im}\delta)^{\perp}\\
A_5^k(M) &= \ker\mathrm{d} \cap \operatorname{im}\delta \cap (\operatorname{im}\mathrm{d})^{\perp}\\
A_6^k(M) &= \operatorname{im}\mathrm{d} \cap \operatorname{im}\delta.
\end{aligned}$$

Proof. From mutual orthogonality between $A_l^k(M)$ we obtain

$$\begin{aligned}
\operatorname{im} \mathrm{d} &= A_1^k(M) \oplus A_4^k(M) \oplus A_6^k(M) \\
\operatorname{im} \delta &= A_2^k(M) \oplus A_5^k(M) \oplus A_6^k(M) \\
\ker \mathrm{d} &= A_1^k(M) \oplus A_3^k(M) \oplus A_4^k(M) \oplus A_5^k(M) \oplus A_6^k(M) \\
\ker \delta &= A_2^k(M) \oplus A_3^k(M) \oplus A_4^k(M) \oplus A_5^k(M) \oplus A_6^k(M).
\end{aligned}$$

Using corollary 3.1, the result now follows from basic set operations. □

Remark 3.2. A special case of proposition 3.1 is given in [11]. Indeed, that paper gives a characterization of the Helmholtz decomposition of vector fields on bounded domains of $\mathbb{R}^3$ in terms of the kernel and image of the grad and curl operators.

3.2. Special case of 2-manifolds

In this section we analyse in detail the complete Hodge decomposition in the case of 2-manifolds. The de Rham complex and co-complex for a Riemannian 2-manifold (M, g) is

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "\mathrm{d}"] \arrow[d, "\star"'] & \Omega^1(M) \arrow[r, "\mathrm{d}"] \arrow[d, "\star"'] & \Omega^2(M) \arrow[d, "\star"'] \\
\Omega^2(M) \arrow[r, "\delta"] & \Omega^1(M) \arrow[r, "\delta"] & \Omega^0(M)
\end{tikzcd}$$

so the Hodge star maps $\Omega^1(M)$ isomorphically to itself. If $\alpha \in \Omega^1(M)$, then $\star\star\alpha = -\alpha$, so the Hodge star induces an almost complex structure on M. Also, since the Hodge star maps normal forms to tangential forms, closed forms to co-closed forms, and exact forms to co-exact forms (and vice-versa), it holds that $\star A_1^1(M) = A_2^1(M)$ and $\star A_4^1(M) = A_5^1(M)$. It also holds that $\star A_3^1(M) = A_3^1(M)$ and $\star A_6^1(M) = A_6^1(M)$.

We now work out the complete Hodge decomposition of $\Omega^1(M)$ in some standard examples. These are relevant later in the paper, when we discuss the Lie algebra of conformal vector fields.

Example 3.1 (disc)**.** Let $\mathbb{D}$ be the unit disc in $\mathbb{R}^2$, equipped with the Euclidean metric. Since the first co-homology group of $\mathbb{D}$ is trivial, it holds that every closed 1-form is exact, and every co-closed 1-form is co-exact. Thus, $A_3^1(\mathbb{D})$, $A_4^1(\mathbb{D})$ and $A_5^1(\mathbb{D})$ are trivial, so all harmonic 1-fields are in $A_6^1(\mathbb{D})$. Thus, the complete Hodge decomposition for $\Omega^1(\mathbb{D})$ is

$$\Omega^1(\mathbb{D}) = A_1^1(\mathbb{D}) \oplus A_2^1(\mathbb{D}) \oplus A_6^1(\mathbb{D}), \tag{2}$$

where all the components are infinite dimensional.

Example 3.2 (annulus)**.** Let $\mathbb{A}$ be a standard annulus in $\mathbb{R}^2$, equipped with the Euclidean metric. As for the disc, it holds that $A_3^1(\mathbb{A})$ is trivial. However, $A_4^1(\mathbb{A})$ and $A_5^1(\mathbb{A})$ are not trivial. Indeed, the tangential 1-form $\mathrm{d}\theta$, where $\theta = \tan(x/y)$ is closed and co-exact, but not exact since θ is not smooth. This is in fact the only non-exact harmonic field, so $A_5^1(\mathbb{A}) = \operatorname{span}\{\mathrm{d}\theta\}$. This also shows that $A_4^1(\mathbb{A}) = \operatorname{span}\{\star\mathrm{d}\theta\} = \operatorname{span}\{\mathrm{d}r/r\}$, where $r^2 = x^2 + y^2$. Thus, the complete Hodge decomposition is

$$\Omega^1(\mathbb{A}) = A_1^1(\mathbb{A}) \oplus A_2^1(\mathbb{A}) \oplus A_4^1(\mathbb{A}) \oplus A_5^1(\mathbb{A}) \oplus A_6^1(\mathbb{A}), \tag{3}$$

where $A_4^1(\mathbb{A})$ and $A_5^1(\mathbb{A})$ are one-dimensional, and all the other components are infinite dimensional.

Example 3.3 (torus)**.** Let $\mathbb{T}^2$ be the flat 2-torus. Since $\mathbb{T}^2$ is a closed manifold, $A_4^1(\mathbb{T}^2)$, $A_5^1(\mathbb{T}^2)$ and $A_6^1(\mathbb{T}^2)$ are trivial. However, the first co-homology group is two dimensional, so $A_3^1(\mathbb{T}^2)$ is also two dimensional. Thus, the complete Hodge decomposition for $\Omega^1(\mathbb{T}^2)$ is

$$\Omega^1(\mathbb{T}^2) = A_1^1(\mathbb{T}^2) \oplus A_2^1(\mathbb{T}^2) \oplus A_3^1(\mathbb{T}^2). \tag{4}$$

Example 3.4 (sphere)**.** Let S^2 be the sphere, equipped with the metric inherited from $\mathbb{R}^3$. Since S^2 is a closed manifold, $A_4^1(S^2)$, $A_5^1(S^2)$ and $A_6^1(S^2)$ are trivial. Since the first co-homology group of S^2 is trivial, it holds that $A_3^1(S^2)$ is trivial. Thus, the complete Hodge decomposition for $\Omega^1(S^2)$ is

$$\Omega^1(S^2) = A_1^1(S^2) \oplus A_2^1(S^2). \tag{5}$$

4. Orthogonal decomposition of vector fields

The L^2 inner product on $\mathfrak{X}(M)$ is given by

$$\langle \xi, \eta \rangle_M := \int_M \mathsf{g}(\xi, \eta)\mathrm{vol}$$

where vol is the volume form induced by g. (Notice that we use the same notation as for the L^2 inner product on forms.)

In this section we show how the Hodge decomposition can be used to obtain the L^2 orthogonal complement of the Lie subalgebras of vector field discussed in section 2 above. The approach is to find an isometry $\mathfrak{X}(M) \to \Omega^1(M)$ which maps the vector field subalgebra under study onto one of the components in the Hodge decomposition.

For the first case of volume preserving vector fields is well known that $\mathfrak{X}(M)$ can be orthogonally decomposed into divergence free plus gradient vector fields. Using contraction with the metric as an isometric isomorphism $\mathfrak{X}(M) \to \Omega^1(M)$, this decomposition is expressed by the Hodge decomposition of 1-forms. (Equivalently, one may use contraction with the volume form instead, which corresponds to the Hodge decomposition of $(n-1)$-forms.) The decomposition is essential in the derivation of the Euler equations for the motion of an ideal incompressible fluid. In this case, the Lagrange multipliers in the projection has the physical interpretation as pressure in the fluid.

For the second case, it is well known that symplectic vector fields can be identified with closed 1-forms, by contraction with the symplectic form. Usually, flow equations for symplectic vector fields are written in terms of the Hamiltonian function, i.e., $\mathfrak{X}_\omega(M)$ is identified with $\mathcal{F}(M)$, and the equations are expressed on $\mathcal{F}(M)$ (an example is the equation for quasigeostrophic motion). This approach, corresponding to vorticity formulation in the case of the Euler fluid, is viable in the setting of Hamiltonian vector fields, i.e., those which have a globally defined Hamiltonian. On manifolds which are not simply connected (so that not every closed 1-form is exact), this approach may not be feasible. However, representation on the full space of vector fields, using Lagrangian multipliers for orthogonal projection, can always be used.

The third case of conformal vector fields is a new example. For a manifold of dimension larger than 2, it follows from a theorem by Liouville that the space of conformal vector fields is finite dimensional. However, in the case of 2-manifolds, the space of conformal vector fields can be (but does not have to be) infinite dimensional. The approach we follow in this paper works in the case of flat 2-manifolds. For closed manifolds, this includes essentially only the 2-torus, for which the set of conformal vector fields is finite dimensional. However, in the case of a bounded domain of $\mathbb{R}^2$, the conformal vector fields correspond to all holomorphic functions on this domain, and is thus infinite dimensional. This is the space of main interest for us, as it is the phase space for the conformal variational problems in section 5.

Each of the vector field subalgebras discussed in section 2 can be identified with one or several components in the complete Hodge decomposition. However, the choice of isomorphism $\mathfrak{X}(M) \to \Omega^1(M)$ is different between the three basic cases of divergence free, symplectic, and conformal vector fields. Once the isomorphism has been specified, we use the short-hand notation $\xi \mapsto \xi^\flat$ for the map $\mathfrak{X}(M) \to \Omega^1(M)$ and $\alpha \mapsto \alpha^\sharp$ for its inverse. Table 1 contains an overview of the decompositions. Detailed expositions for each case are given in the remaining part of this section.

Remark 4.1. The requirement that the isomorphism $\mathfrak{X}(M) \to \Omega^1(M)$ is an isometry can be weakened. Indeed, our basic requirement is that orthogonality is preserved, so it is enough that the map is conformal. However, in the examples in this paper the isomorphism will be an isometry.

4.1. Volume preserving vector fields on a Riemannian manifold

Let (M, g) be a compact Riemannian manifold. As isomorphism we use contraction with the metric, i.e., $\xi \mapsto \mathrm{i}_\xi \mathsf{g} =: \xi^\flat$.

As before, let vol denote the volume form induces by the Riemannian metric g. Since ξ is divergence free if and only if $\mathrm{i}_\xi \mathrm{vol}$ is closed, it follows from the formula $\mathrm{i}_\xi \mathrm{vol} = \star\xi^\flat$ that ξ is divergence free if and only if $\xi^\flat$ is co-closed, i.e., $\delta\xi^\flat = 0$. Furthermore, $\xi \in \mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)$ if and only if $\xi^\flat$ is co-exact, i.e., $\xi^\flat = \delta\alpha$ for some $\alpha \in \Omega^2(M)$. Thus, $\mathfrak{X}_{\mathrm{vol}}(M)^\flat = \ker\delta$ and $\mathfrak{X}^{\mathrm{ex}}_{\mathrm{vol}}(M)^\flat = \mathrm{im}\,\delta$, which gives the first and the third decompositions in table 1.

For the case of tangential vector fields, we notice that ξ is a tangential vector field if and only if $\xi^\flat$ is a tangential 1-form. Since elements in $A^1_4(M)$ and $A^1_6(M)$ are necessarily non-tangential, we obtain the second and the fourth decompositions in table 1.

4.2. Symplectic vector fields on an almost Kähler manifold

Let (M, g, ω) be an almost Kähler manifold. As isomorphism we use contraction with the symplectic form, i.e., $\xi \mapsto \mathrm{i}_\xi\omega =: \xi^\flat$. Due to the almost Kähler structure, this isomorphism is isometric.

Since $\pounds_\xi\omega = \mathrm{d}\mathrm{i}_\xi\omega = \mathrm{d}\xi^\flat$ it follows that ξ is symplectic if and only if $\xi^\flat$ is closed. Likewise, ξ is Hamiltonian if and only if $\xi^\flat$ is exact. Thus, $\mathfrak{X}_\omega(M)^\flat = \ker\mathrm{d}$ and $\mathfrak{X}_{\mathrm{Ham}}(M)^\flat = \mathrm{im}\,\mathrm{d}$. This yields the first and the third symplectic decompositions in table 1.

For the case of tangential symplectic and Hamiltonian vector fields, we notice that ξ is a tangential vector field if and only if $\xi^\flat$ is a normal 1-form. Since elements in $A^1_5(M)$ and $A^1_6(M)$ are necessarily non-tangential, we obtain the second and the fourth symplectic decompositions in table 1.

4.3. Conformal vector fields on a 2-manifold

Let (M, g) be a compact Riemannian 2-manifold, possibly with boundary. If (M, g) is flat, then there exists an orthogonal reflection map $R \in \mathcal{T}^1_1(M)$, i.e., an orientation reversing isometry such that $R^2 = \mathsf{Id}$. In that case, we may chose coordinate charts such that $\mathsf{g} = \mathrm{d}x\otimes\mathrm{d}x + \mathrm{d}y\otimes\mathrm{d}y$ and $R = \mathrm{d}x \otimes \partial_x - \mathrm{d}y \otimes \partial_y$. If $\xi \in \mathfrak{X}(M)$ then we write $\bar{\xi} := R\xi$. Since R is an isometry, it holds that the isomorphism

$$\mathfrak{X}(M) \ni \xi \mapsto \mathrm{i}_{\bar{\xi}}\mathsf{g} =: \xi^\flat \in \Omega^1(M)$$

is isometric with respect to the L^2 inner products on $\mathfrak{X}(M)$ and $\Omega^1(M)$. The following lemma is the key to obtaining the L^2 orthogonal complement of $\mathfrak{X}_{\text{con}}(M)$ in $\mathfrak{X}(M)$.

Lemma 4.1. *If* (M, g) *is a flat 2-manifold then* $\mathfrak{X}_{\text{con}}(M)^\flat = \mathcal{H}^1(M)$.

Proof. It is enough to prove the assertion in local coordinates as above. Let $\xi = u\partial_x + v\partial_y \in \mathfrak{X}_{\text{con}}(M)$. Then (u, v) must fulfil the Cauchy–Riemann equations.

It holds that $\xi^\flat = u\,\mathrm{d}x - v\,\mathrm{d}y$. Thus,

$$\mathrm{d}\xi^\flat = u_y\,\mathrm{d}y \wedge \mathrm{d}x - v_x\,\mathrm{d}x \wedge \mathrm{d}y = -(v_x + u_y)\,\mathrm{d}x \wedge \mathrm{d}y$$

and

$$\delta\xi^\flat = \star\,\mathrm{d}\star\xi^\flat = \star\,\mathrm{d}(v\,\mathrm{d}x + u\,\mathrm{d}y) = \star(u_x - v_y)\,\mathrm{d}x \wedge\,\mathrm{d}y = u_x - v_y.$$

Hence, we see that $\mathrm{d}\xi^\flat = 0$ and $\delta\xi^\flat = 0$ if and only if (u, v) fulfils the Cauchy–Riemann equations. This proves the assertion. □

We now introduce 'reflected' versions of the gradient and the skew gradient. Indeed, for a function $F \in \mathcal{F}(M)$ we define the *reflection gradient* as $\overline{\text{grad}}(F) = (\mathrm{d}F)^\sharp$ and the *reflection skew gradient* as $\overline{\text{sgrad}}(F) = (\star\,\mathrm{d}F)^\sharp$. In local (flat) coordinates we have

$$\overline{\text{grad}}(F) = F_x\partial_x - F_y\partial_y, \quad \overline{\text{sgrad}}(F) = F_y\partial_x + F_x\partial_y.$$

Let $\mathcal{F}_0(M) = \{F \in \mathcal{F}(M); F|_{\partial M} = 0\}$, i.e., the functions that vanish on the boundary. Using lemma 4.1 together with the Hodge decomposition theorem 3.1 we obtain the following result.

Proposition 4.1. *If* (M, g) *is a flat 2-manifold, then the space of vector fields on M admits the* L^2 *orthogonal decomposition*

$$\mathfrak{X}(M) = \mathfrak{X}_{\text{con}}(M) \oplus \overline{\text{grad}}(\mathcal{F}_0(M)) \oplus \overline{\text{sgrad}}(\mathcal{F}_0(M))$$

where each component is closed in $\mathfrak{X}(M)$ *with respect to the Fréchet topology.*

Proof. First, we need to verify the following diagram:

$$\begin{array}{ccccc}
\overline{\text{grad}}(\mathcal{F}_0(M)) & \oplus\ \overline{\text{sgrad}}(\mathcal{F}_0(M)) & \oplus & \mathfrak{X}_{\text{con}}(M) \\
\downarrow{\scriptstyle\flat} & \downarrow{\scriptstyle\flat} & & \downarrow{\scriptstyle\flat} \\
\mathrm{d}\Omega^0_{\mathsf{n}}(M) \quad \oplus & \delta\Omega^2_{\mathsf{t}}(M) & \oplus & \mathcal{H}^1(M)
\end{array}$$

It follows from the definition of normal forms that $\Omega^0_{\mathsf{n}}(M) = \mathcal{F}_0(M)$. Thus,

$$\overline{\text{grad}}(\mathcal{F}_0(M))^\flat = \mathrm{d}\mathcal{F}_0(M) = \mathrm{d}\Omega^0_{\mathsf{n}}(M).$$

Also, it holds that $\Omega^0_{\mathsf{n}}(M) = \star\Omega^2_{\mathsf{t}}(M)$, which gives

$$\overline{\text{sgrad}}(\mathcal{F}_0(M))^\flat = \star\,\mathrm{d}\Omega^0_{\mathsf{n}}(M) = -\delta\Omega^2_{\mathsf{t}}(M) = \delta\Omega^2_{\mathsf{t}}(M).$$

Next, $\mathfrak{X}_{\text{con}}(M)^\flat = \mathcal{H}^1(M)$ follows from lemma 4.1. Now, from the Hodge decomposition of forms (theorem 3.1) it follows that the subspaces of forms are L^2 orthogonal to each other. This implies that the subspaces of vector fields are also L^2 orthogonal to each other, since the isomorphism $\xi \to \xi^\flat$ is an isometry. Finally, since the isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is a continuous (even smooth) vector space isomorphism with respect to the Fréchet topologies on $\mathfrak{X}$ and $\Omega^1(M)$, and since the subspaces of forms are topologically closed in $\Omega^1(M)$, it follows that the vector field subspaces are topologically closed in $\mathfrak{X}(M)$. □

Example 4.1 (disc)**.** Let $M = \mathbb{D}$ and let (x, y) be Cartesian coordinates. If $\xi = u\partial_x + v\partial_y$, we take as reflection map $R(\xi) = u\partial_x - v\partial_y$ (corresponding to complex conjugation of $z = x + \mathrm{i}y$). It follows from (2) that $\mathcal{H}^1(\mathbb{D}) = A^1_6(\mathbb{D})$. Thus, by proposition 4.1 we get that the set of conformal vector fields on $\mathbb{D}$ (corresponding to holomorphic functions on $\mathbb{D}$) is isometrically isomorphic to the set of simultaneously exact and co-exact 1-forms.

Example 4.2 (annulus)**.** Let $M = \mathbb{A}$ and let (x, y) be Cartesian coordinates. We use the same reflection map as for the disc. It follows from (3) that $\mathcal{H}^1(\mathbb{D}) = A^1_4(\mathbb{D}) \oplus A^1_5(\mathbb{D}) \oplus A^1_6(\mathbb{D})$. The special harmonic fields $\mathrm{d}\theta$ and $\mathrm{d}r/r$ corresponds to the holomorphic functions i/z and $1/z$. In complex analysis it is well known that these functions have a special role (e.g., in the calculus of residues).

Example 4.3 (torus)**.** Let $M = \mathbb{T}$ and let (θ, ϕ) be the standard angle coordinates. (Notice that θ, ϕ are not smooth functions on $\mathbb{T}$, but $\mathrm{d}\theta$ and $\mathrm{d}\phi$ are well defined smooth 1-forms.) We use the reflection map $(\partial_\theta, \partial_\phi) \mapsto (\partial_\theta, -\partial_\phi)$. It follows from (4) and proposition 4.1 that the set of conformal vector fields is only two dimensional, generated by the pure translations ∂_θ and ∂_ϕ.

Counter-example 4.4 (sphere)**.** It follows from (5) that the space of harmonic fields on the sphere is trivial. However, the space of conformal vector fields of the sphere corresponds to the Möbius transformations (by identifying the plane with the Riemann sphere) and is thus six dimensional, see e.g. [12]. Thus, it is *not* possible to identify the space of conformal vector fields with the harmonic 1-fields in this case. The reason is that S^2 is not flat, so proposition 4.1 does not apply.

5. Conformal variational problems

Using the decomposition of conformal vector fields derived in subsection 4.3, we are now ready to tackle our original problem: to derive differential equations for variational problems involving conformal vector fields on a bounded domain $U \subset \mathbb{R}^2$ (or equivalently, variational problems involving holomorphic functions on a bounded complex domain U). The two remaining tools we need involve (i) computing the conformal projection and (ii) integration by parts. We consider these first.

5.1. Computing the conformal projection

Proposition 5.1. *Let U be a bounded domain in $\mathbb{R}^2$. Then the orthogonal projection $\mathrm{Pr}_{\mathrm{con}} : \mathfrak{X}(U) \to \mathfrak{X}_{\mathrm{con}}(U)$ of a vector field onto its conformal component corresponding to the orthogonal decomposition in theorem 4.1 is given by*

$$\mathrm{Pr}_{\mathrm{con}}(f\partial_x + g\partial_y) = f\partial_x + g\partial_y - \overline{\mathrm{grad}}(F) - \overline{\mathrm{sgrad}}(G)$$

where F and G satisfy the Poisson equations

$$\begin{aligned} \nabla^2 F &= f_x - g_y, \quad F|_{\partial U} = 0 \\ \nabla^2 G &= f_y + g_x, \quad G|_{\partial U} = 0. \end{aligned} \tag{6}$$

Proof. Compute $f_x - g_y$ and $f_y + g_x$ using the orthogonal decomposition in proposition 4.1. □

The Poisson equations (6) are analogous to the Poisson equation that determines the pressure when projecting a vector field to its volume preserving component. Note that F and G vanish (as they must) if $f\partial_x + g\partial_y$ satisfies the Cauchy–Riemann equations.

There is also a direct way to compute $\mathrm{Pr}_{\mathrm{con}}$ using the *Bergman kernel* [13], the reproducing kernel $K_U(z,\cdot)$ of the Bergman space $A^2(U)$, which is the Hilbert space obtained by completion of $\mathfrak{X}_{\mathrm{con}}(U)$ with respect to the L^2 inner product. Recall that a reproducing kernel on a Hilbert space of functions $U \to \mathbb{C}$ is a mapping $K : U \times U \to \mathbb{C}$ such that

$$f(x) = \langle\!\langle f, K(x,\cdot)\rangle\!\rangle$$

for all functions f belonging to the Hilbert space. The existence of the reproducing kernel $K_U(z,\cdot)$ for the Hilbert space $A^2(U)$ follows from Riesz representation theorem, since point-wise evaluation of functions in $A^2(U)$ is linear and continuous in the L^2 topology.

Proposition 5.2. *Let $K_U(z,\cdot)$ be the reproducing kernel of $A^2(U)$. Then for any complex-valued function $\xi \in L^2(U)$, $\mathrm{Pr}_{\mathrm{con}}(\xi) = \langle\!\langle \xi, K_U(z,\cdot)\rangle\!\rangle_U$ where $\langle\!\langle f, g\rangle\!\rangle_U = \int_U f\bar{g}\,\mathrm{d}A$ is the complex L^2 inner product. For $U = \mathbb{D}$,*

$$\mathrm{Pr}_{\mathrm{con}}\left(\sum_{m,n=0}^{\infty} a_{mn} z^m \bar{z}^n\right) = \sum_{m\geqslant n\geqslant 0} a_{mn}\frac{m-n+1}{m+1} z^{m-n}.$$

Proof. For any $f \in A^2(U)$ it then holds that $f(z) = \langle\!\langle f, K_U(z,\cdot)\rangle\!\rangle_U$. The complex and real L^2 inner products are related by $\langle\!\langle f, g\rangle\!\rangle_U = \langle f, g\rangle_U - \mathrm{i}\langle \mathrm{i}f, g\rangle_U$, so $\langle\!\langle \mathfrak{X}_{\mathrm{con}}(U)^{\perp}, \mathfrak{X}_{\mathrm{con}}(U)\rangle\!\rangle = 0$. Therefore, for $\xi \in \mathfrak{X}(U)$ we have

$$\mathrm{Pr}_{\mathrm{con}}(\xi) = \langle\!\langle \mathrm{Pr}_{\mathrm{con}}(\xi), K_U(z,\cdot)\rangle\!\rangle_U = \langle\!\langle \xi, K_U(z,\cdot)\rangle\!\rangle_U,$$

because $\mathrm{Pr}_{\mathrm{con}}(\xi) - \xi$ is orthogonal to all holomorphic functions, in particular to $\overline{K_U(z,w)}$ which is holomorphic in w.

For the unit disc, $K_{\mathbb{D}}(z,\zeta) = \frac{1}{\pi}\frac{1}{(1-\bar{z}\zeta)^2}$. A basis for $\mathfrak{X}(\mathbb{D})$ is given by $\{z^m\bar{z}^n\}_{m,n\geqslant 0}$. Thus, if $\xi \in \mathfrak{X}(\mathbb{D})$ is expanded in this basis, we may compute $\mathrm{Pr}_{\mathrm{con}}(\xi)$ by applying $\mathrm{Pr}_{\mathrm{con}}$ to each of the basis elements. Indeed, if $e_{mn}(z) = z^m\bar{z}^n$ then

$$\begin{aligned}
\mathrm{Pr}_{\mathrm{con}}(e_{mn})(z) &= \int_{\mathbb{D}} \frac{1}{\pi}\frac{1}{(1-z\bar{\zeta})^2}\zeta^m\bar{\zeta}^n\,\mathrm{d}A(\zeta)\\
&= \int_{\mathbb{D}} \frac{1}{\pi}\sum_{p=1}^{\infty} z^p\bar{\zeta}^p\zeta^m\bar{\zeta}^n\,\mathrm{d}A(\zeta)\\
&= \begin{cases} \frac{m-n+1}{m+1} z^{m-n} & m \geqslant n\\ 0 & m < n. \end{cases}
\end{aligned}$$

□

For other domains, $K_U(z,\zeta) = K_{\mathbb{D}}(\varphi(z),\varphi(\zeta))\varphi'(\zeta)\overline{\varphi'(z)}$, where φ is a conformal mapping $U \to \mathbb{D}$. Another result that may be useful is the following.

Proposition 5.3. *Let $\xi \in \mathfrak{X}(U)$ and $\psi \in \mathfrak{X}_{\mathrm{con}}(U)$. If $\psi(z) \neq 0$ for all $z \in U$ then $\mathrm{Pr}_{\mathrm{con}}(\xi) = 0$ if and only if $\mathrm{Pr}_{\mathrm{con}}(\bar{\psi}\xi) = 0$.*

Proof. We have

$$\begin{aligned}
\mathrm{Pr}_{\mathrm{con}}(\xi) = 0 &\iff \langle \xi, \eta\rangle = 0 && \forall \eta \in \mathfrak{X}_{\mathrm{con}}(U)\\
&\iff \langle \bar{\psi}\xi, \eta/\psi\rangle = 0 && \forall \eta \in \mathfrak{X}_{\mathrm{con}}(U)\\
&\iff \langle \bar{\psi}\xi, \rho\rangle = 0 && \forall \rho \in \mathfrak{X}_{\mathrm{con}}(U)\\
&\iff \mathrm{Pr}_{\mathrm{con}}(\bar{\psi}\xi) = 0.
\end{aligned}$$

□

5.2. Integration by parts

Standard variational calculus makes frequent use of integration by parts in order to 'isolate' a virtual variation from derivatives. Usually, the boundary term appearing either vanishes (in the case of a space of tangential vector fields), or it can be treated separately giving rise to natural boundary conditions (in the case of a space where vector fields can have arbitrary small compact support). However, in the case of conformal vector fields, there is always a global dependence between interior points and points on the boundary (due to the Cauchy–Riemann equations). Hence, in the conformal case, we need an appropriate analogue of integration by parts which avoids boundary integrals. Formally, we may proceed as follows. Let $\partial_z : \mathfrak{X}_{\mathrm{con}}(U) \to \mathfrak{X}_{\mathrm{con}}(U)$ be the complex derivative. Then we are looking for the adjoint of this operator with respect to the L^2 inner product. That is, an operator $\partial_z^\top : \mathfrak{X}_{\mathrm{con}}(U) \to \mathfrak{X}_{\mathrm{con}}(U)$ such that

$$\langle \xi, \partial_z \eta \rangle_U = \langle \partial_z^\top \xi, \eta \rangle_U, \quad \forall\, \xi, \eta \in \mathfrak{X}_{\mathrm{con}}(U).$$

This operator can be evaluated explicitly on the unit disc and on the image of a given conformal map $\varphi : \mathbb{D} \to U$.

Proposition 5.4. *(i) For $\xi, \eta \in \mathfrak{X}_{\mathrm{con}}(\mathbb{D})$, denoting $\partial_z \xi = \xi_z$, we have*

$$\langle \xi, \eta_z \rangle_{\mathbb{D}} = \langle (z^2 \xi)_z, \eta \rangle_{\mathbb{D}}.$$

(ii) For $\xi, \eta \in \mathfrak{X}_{\mathrm{con}}(\varphi(\mathbb{D}))$, there are holomorphic functions

$$\chi_1 := (z^2 \varphi_z)_z \circ \varphi^{-1}, \quad \chi_2 := (z^2 \varphi_z^2) \circ \varphi^{-1}$$

on $\varphi(\mathbb{D})$ such that

$$\langle \xi, \eta_z \rangle_{\varphi(\mathbb{D})} = \langle |\varphi_z|^{-2} (\chi_1 \xi + \chi_2 \xi_z, \eta \rangle$$

and

$$\partial_z^\top \xi = \mathrm{Pr}_{\mathrm{con}}\big(|\varphi_z \circ \varphi^{-1}|^{-2} (\chi_1 \xi + \chi_2 \xi_z) \big).$$

Proof. (i) Every element of $\mathfrak{X}_{\mathrm{con}}(\mathbb{D})$ has a convergent Taylor series, and the monomials z^n form a basis for $\mathfrak{X}_{\mathrm{con}}(\mathbb{D})$ that is orthogonal with respect to both the real and the complex L^2 inner product, for

$$\begin{aligned}\langle\!\langle z^m, z^n \rangle\!\rangle_{\mathbb{D}} &= \int_{\mathbb{D}} \xi \bar{\eta}\, \mathrm{d}A \\ &= \int_0^1 \left(r\, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, r^m\, \mathrm{e}^{\mathrm{i}m\theta} r^n\, \mathrm{e}^{-\mathrm{i}n\theta} \right) \\ &= \frac{2\pi}{m+n+2} \delta_{m,n}\end{aligned}$$

and $\langle \xi, \eta \rangle_{\mathbb{D}} = \mathrm{Re}\, \langle\!\langle \xi, \eta \rangle\!\rangle_{\mathbb{D}}$. Therefore, expanding $\xi = \sum_{n=0}^\infty \xi_n z^n$ and $\eta = \sum_{m=0}^\infty \eta_m z^m$,

$$\begin{aligned}\langle\!\langle \xi, \eta_z \rangle\!\rangle_{\mathbb{D}} &= \left\langle\!\!\left\langle \sum_{n=0}^\infty \xi_n z^n, \left(\sum_{m=0}^\infty \eta_m z^m \right)_z \right\rangle\!\!\right\rangle_{\mathbb{D}} \\ &= \sum_{n,m=0}^\infty \xi_n \bar{\eta}_m \langle\!\langle z^n, m z^{m-1} \rangle\!\rangle_{\mathbb{D}} \\ &= \sum_{n,m=0}^\infty \xi_n \bar{\eta}_m \frac{2\pi m}{n+m+1} \delta_{n,m-1} \\ &= \sum_{n=0}^\infty \pi \xi_n \bar{\eta}_{n+1}\end{aligned}$$

while

$$\begin{aligned}\langle\!\langle (z^2\xi)_z, \eta\rangle\!\rangle_{\mathbb{D}} &= \sum_{n,m=0}^{\infty} \xi_n \bar{\eta}_m \langle\!\langle (n+2)z^{n+1}, z^m\rangle\!\rangle_{\mathbb{D}}\\ &= \sum_{n,m=0}^{\infty} \xi_n \bar{\eta}_m (n+2)\frac{2\pi}{n+m+3}\delta_{n+1,m}\\ &= \sum_{n=0}^{\infty} \pi \xi_n \bar{\eta}_{n+1}.\end{aligned}$$

Since the two complex inner products are equal, their real parts are equal, establishing the result. (ii) We have

$$\begin{aligned}\langle \xi, \eta_z\rangle_{\varphi(\mathbb{D})} &= \langle \varphi_z \xi\circ\varphi, \varphi_z\eta_z\circ\varphi\rangle_{\mathbb{D}}\\ &= \langle \varphi_z\xi\circ\varphi, (\eta\circ\varphi)_z\rangle_{\mathbb{D}}\\ &= \langle (z^2\varphi_z\xi\circ\varphi)_z, \eta\circ\varphi\rangle_{\mathbb{D}}\\ &= \langle (\varphi_z\circ\varphi^{-1})^{-1}(z^2\varphi_z\xi\circ\varphi)_z)\circ\varphi^{-1}, (\varphi_z\circ\varphi^{-1})^{-1}\eta\rangle_{\varphi(\mathbb{D})}\\ &= \langle (|\varphi_z\circ\varphi^{-1}|^{-2}(z^2\varphi_z\xi\circ\varphi)_z)\circ\varphi^{-1}, \eta\rangle_{\varphi(\mathbb{D})}\\ &= \langle |\varphi_z\circ\varphi^{-1}|^{-2}(\chi_1\xi+\chi_2\xi_z), \eta\rangle_{\varphi(\mathbb{D})}.\end{aligned}$$

□

The Hodge decomposition for conformal vector fields, the projection to conformal vector fields, and the integration by parts result now allow the calculation of the equations of motion for any Lagrangian density on $\mathfrak{X}_{\mathrm{con}}(\varphi(\mathbb{D}))$. Before we come to the conformal flow equation in subsection 5.5, we give two basic examples in subsections 5.3 and 5.4.

5.3. Example (Conformal stationary problem)

Let $V \in \mathcal{F}(\mathbb{R}^2)$ and consider the Lagrangian density $\mathcal{L}(\xi,\xi_z) = \frac{1}{2}|\xi_z|^2 + V(\xi)$. Let $S : \mathfrak{X}_{\mathrm{con}}(U) \to \mathbb{R}$ be the corresponding action $S(\xi) = \int_U \mathcal{L}(\xi(z), \xi_z(z))\, \mathrm{d}A(z)$ and consider the variational problem:

$$\text{Find } \xi \in \mathfrak{X}_{\mathrm{con}}(U) \text{ such that } \frac{\delta S}{\delta\xi}(\xi)\cdot\eta = 0 \text{ for all variations } \eta\in\mathfrak{X}_{\mathrm{con}}(U).$$

Direct calculations yield

$$\begin{aligned}\frac{\delta S}{\delta\xi}(\xi)\cdot\eta &= \langle \xi_z, \eta_z\rangle_U + \langle \mathrm{grad}(V)\circ\xi, \eta\rangle_U\\ &= \langle \partial_z^{\top}\xi_z, \eta\rangle_U + \langle \mathrm{grad}(V)\circ\xi, \eta\rangle_U\\ &= \langle \partial_z^{\top}\xi_z + \mathrm{grad}(V)\circ\xi, \eta\rangle_U.\end{aligned}$$

We require this to vanish for all $\eta\in\mathfrak{X}_{\mathrm{con}}(\mathbb{D})$. That is, the first term in the inner product must be orthogonal to all conformal vector fields, i.e.,

$$\mathrm{Pr}_{\mathrm{con}}(\partial_z^{\top}\xi_z + \mathrm{grad}(V)\circ\xi) = 0.$$

Since $\partial_z^{\top}\xi_z$ is already holomorphic we get

$$\partial_z^{\top}\xi_z + \mathrm{Pr}_{\mathrm{con}}(\mathrm{grad}(V)\circ\xi) = 0.$$

Now, using the orthogonal decomposition of conformal vector fields, derived in subsection 2.3 above, we introduce Lagrange multipliers $F, G \in \mathcal{F}_0(U)$ for the constraints, giving the differential equation

$$\begin{aligned}
&\partial_z^\top \xi_z + \operatorname{grad}(V)\circ\varphi = \overline{\operatorname{grad}}(F) + \overline{\operatorname{sgrad}}(G)\\
&\frac{\partial \xi}{\partial \bar z} = 0\\
&F|_{\partial U} = G|_{\partial U} = 0
\end{aligned}$$

where $\frac{\partial \xi}{\partial \bar z} = 0$ is short way to write the Cauchy–Riemann equations. Notice that it is not certain that the original variational problem is well-posed. That requires additional assumptions on the function V.

5.4. Example (conformal wave equation)

Adding time to the previous example, and denoting $\frac{\mathrm{d}}{\mathrm{d}t}\xi = \xi_t$, we consider the Lagrangian density corresponding to a nonlinear conformal wave equation

$$\mathcal{L}(\xi, \xi_z, \xi_t) = \tfrac{1}{2}|\xi_t|^2 - \tfrac{1}{2}|\xi_z|^2 - V(\xi).$$

Requiring that the action $S(\xi) = \int_0^1 \int_U \mathcal{L}(\xi(z,t), \xi_z(z,t))\, \mathrm{d}A(z)\, \mathrm{d}t$ be stationary on paths in $\mathfrak{X}_{\mathrm{con}}(U)$ fixed at the initial and final times yields

$$\langle -\xi_{tt} - \partial_z^\top \xi_z - \operatorname{grad}(V)\circ\xi, \eta\rangle_U = 0$$

and so the equations of motion are

$$\xi_{tt} + \partial_z^\top \xi_z + \mathrm{Pr}_{\mathrm{con}}(\operatorname{grad}(V)\circ\xi) = 0,$$

or, using Lagrangian multipliers,

$$\begin{aligned}
&\xi_{tt} + \partial_z^\top \xi_z + \operatorname{grad}(V)\circ\xi = \overline{\operatorname{grad}}(F) + \overline{\operatorname{sgrad}}(G),\\
&\frac{\partial \xi}{\partial \bar z} = 0\\
&F|_{\partial U} = G|_{\partial U} = 0.
\end{aligned} \tag{7}$$

Consider now the case $U = \mathbb{D}$ and $V(z) = c|z|^2/2$ for a constant $c \in \mathbb{R}$. This gives the partial differential equation

$$\xi_{tt} + (z^2\xi_z)_z + c\xi = 0.$$

Expanding ξ in the monomial basis $\xi = \sum_{m=0}^{\infty} \xi_m z^m$, we get

$$(\xi_m)_{tt} + (m^2 + m + c)\xi_m = 0,$$

i.e., a set of uncoupled harmonic oscillators. In particular, there are an infinite number of first integrals, given by

$$I_m(\xi, \dot\xi) = \tfrac{1}{2}|\dot\xi_m|^2 + \tfrac{1}{2}(m^2 + m + c)|\xi_m|^2.$$

We can compare the previous equation (7) to the standard wave equation with Lagrangian density $L(\xi, \xi_x, \xi_y, \xi_t) = \frac{1}{2}|\xi_t|^2 - \frac{1}{4}|\xi_x|^2 - \frac{1}{4}|\xi_y|^2 - V(\xi)$ and zero Dirichlet boundary conditions (if ξ is holomorphic, then $L(\xi, \xi_x, \xi_y, \xi_t) = L(\xi, \xi_z, \xi_t)$). As is well known, the equations of motion for $\xi \in \mathfrak{X}(U)$ are

$$\xi_{tt} - \xi_{xx} - \xi_{yy} + \operatorname{grad}(V)\circ\xi = 0, \quad \xi|_{\partial U} = 0.$$

5.5. Example (geodesic conformal flow equation)

As mentioned, this example constitutes the original motivation for the paper. A weak form of the equation was derived by the authors in [4] (the strong form is also stated there, but without derivation). In this section we include all steps, from formulation of the geodesic problem to the presentation of the governing partial differential equations.

Let $\mathrm{Emb}(\mathbb{D}, \mathbb{R}^2)$ denote the set of embeddings $\mathbb{D} \to \mathbb{R}^2$. Thus set has the structure of a Fréchet–Lie manifold. Although this manifold is not a group, it has many similarities with the diffeomorphism group $\mathrm{Diff}(\mathbb{D})$. First of all, we notice that $\mathrm{Diff}(\mathbb{D})$ is a submanifold of $\mathrm{Emb}(\mathbb{D}, \mathbb{R}^2)$. Secondly, it holds that the tangent space at the identity is equal to the set of all vector fields on $\mathbb{D}$, i.e., $T_{\mathrm{Id}}\mathrm{Emb}(\mathbb{D}, \mathbb{R}^2) = \mathfrak{X}(\mathbb{D})$, which, as reviewed earlier, carries the structure of a Fréchet–Lie algebra with the vector field commutator. (Recall that the subalgebra $\mathfrak{X}_{\mathrm{t}}(\mathbb{D})$ of tangential vector fields is the tangent space at the identity of $\mathrm{Diff}(\mathbb{D})$.)

A Riemannian metric on $\mathrm{Emb}(\mathbb{D}, \mathbb{R}^2)$ is given by

$$T_\varphi \mathrm{Emb}(\mathbb{D}, \mathbb{R}^2) \times T_\varphi \mathrm{Emb}(\mathbb{D}, \mathbb{R}^2) \ni (u, v) \longmapsto \langle u \circ \varphi^{-1}, v \circ \varphi^{-1} \rangle_{\varphi(\mathbb{D})} \in \mathbb{R}. \tag{8}$$

Notice that this metric is invariant under the group $\mathrm{Diff}(\mathbb{D})$ acting on $T\,\mathrm{Emb}(\mathbb{D}, \mathbb{R}^2)$ by composition from the right.

Our aim is to derive the geodesic equation with respect to the metric (8) restricted to the submanifold of conformal embeddings

$$\mathrm{Con}(\mathbb{D}, \mathbb{R}^2) = \{\varphi \in \mathrm{Emb}(\mathbb{D}, \mathbb{R}^2); \varphi^*\mathsf{g} = F\mathsf{g}\},$$

where g is the Euclidean metric on $\mathbb{R}^2$, expressed in terms of (φ, ξ) where $\xi := \dot\varphi \circ \varphi^{-1}$ is the spatial (or Eulerian, or right-reduced) velocity field. These are the variables for the spatial representation of free boundary continua considered for the configuration space $\mathrm{Emb}(\mathbb{D}, \mathbb{R}^n)$ in [14], of the EPDiff equation for the configuration space $\mathrm{Diff}(\mathbb{D})$ considered in [5], and of the standard Eulerian representation of incompressible fluids for the configuration space $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{D})$. We shall see that the conformal setting, with corresponding smaller symmetry group, introduces new aspects to the equations of motion.

By right translation, the tangent space at $\varphi \in \mathrm{Con}(\mathbb{D}, \mathbb{R}^2)$ can be identified with a conformal vector field over the domain $\varphi(\mathbb{D})$. Indeed, we have the isomorphism

$$T\mathrm{Con}(\mathbb{D}, \mathbb{R}^2) \ni (\varphi, \dot\varphi) \mapsto (\varphi, \underbrace{\dot\varphi \circ \varphi^{-1}}_{\xi}) \in \mathrm{Con}(\mathbb{D}, \mathbb{R}^2) \times \mathfrak{X}_{\mathrm{con}}(\varphi(\mathbb{D})),$$

where $\mathrm{Con}(\mathbb{D}, \mathbb{R}^2) \times \mathfrak{X}_{\mathrm{con}}(\varphi(\mathbb{D}))$ should be thought of as a vector bundle over $\mathrm{Con}(\mathbb{D}, \mathbb{R}^2)$.

We have the Lagrangian

$$L(\varphi, \dot\varphi) = \tfrac{1}{2}\langle \varphi_t \circ \varphi^{-1}, \varphi_t \circ \varphi^{-1} \rangle_{\varphi(\mathbb{D})}.$$

First, if φ_ε is a variation of a curve $\varphi(t)$ and $\xi_\varepsilon = \dot\varphi_\varepsilon \circ \varphi_\varepsilon^{-1}$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \xi_\varepsilon = \dot\eta + \pounds_\eta \xi,$$

where $\varphi_\varepsilon = \exp(\varepsilon\eta) \circ \varphi$ with $\eta \in \mathfrak{X}_{\mathrm{con}}(\varphi(\mathbb{D}))$, see [14]. In addition, it holds that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \frac{1}{2}\langle \xi, \xi \rangle_{\varphi_\varepsilon(\mathbb{D})} = \langle \pounds_\eta \xi + \mathrm{div}(\eta)\xi, \xi \rangle_{\varphi(\mathbb{D})}.$$

This equality follows by straightforward calculations and the fact that $\pounds_\xi \mathsf{g} = \mathrm{div}(\xi)\mathsf{g}$ for any $\xi \in \mathfrak{X}_{\mathrm{con}}(\varphi(\mathbb{D}))$, as derived in subsection 2.3 above.

Using these relations, the variational principle now yields

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\int_0^1 L(\varphi_\varepsilon,\dot\varphi_\varepsilon)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\int_0^1 \frac{1}{2}\langle\xi_\varepsilon,\xi_\varepsilon\rangle_{\varphi_\varepsilon(\mathbb{D})}\,\mathrm{d}t\\
&= \int_0^1 \langle\dot\eta + 2\pounds_\eta\xi + \operatorname{div}(\eta)\xi,\xi\rangle_{\varphi(\mathbb{D})}\\
&= \int_{\varphi(\mathbb{D})}\Big(\mathsf{g}(\dot\eta,\xi)\mathrm{vol} + \underbrace{\pounds_\eta}_{\mathrm{di}_\eta}\big(\mathsf{g}(\xi,\xi)\mathrm{vol}\big) - \mathsf{g}(\xi,\xi)\underbrace{\operatorname{div}(\eta)\mathrm{vol}}_{\mathrm{di}_\eta\mathrm{vol}}\Big)\\
&= \int_{\varphi(\mathbb{D})}(\mathsf{g}(\dot\eta,\xi)\mathrm{vol} + \mathrm{d}(\mathsf{g}(\xi,\xi)\mathrm{i}_\eta\mathrm{vol}) - \mathrm{d}(\mathsf{g}(\xi,\xi)\mathrm{i}_\eta\mathrm{vol}) + \mathrm{d}\mathsf{g}(\xi,\xi)\wedge\mathrm{i}_\eta\mathrm{vol})\\
&= \int_{\varphi(\mathbb{D})}(\mathsf{g}(\dot\eta,\xi)\mathrm{vol} + \mathrm{d}\mathsf{g}(\xi,\xi))\wedge\mathrm{i}_\eta\mathrm{vol})\\
&= \langle\dot\eta,\xi\rangle_{\varphi(\mathbb{D})} + \langle\operatorname{grad}(|\xi|^2),\eta\rangle_{\varphi(\mathbb{D})}.
\end{aligned}
$$

Next, since

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta,\xi\rangle_{\varphi(\mathbb{D})} &= \langle\dot\xi,\eta\rangle_{\varphi(\mathbb{D})} + \langle\xi,\dot\eta\rangle_{\varphi(\mathbb{D})} + \int_{\varphi(\mathbb{D})}\pounds_\xi(\mathsf{g}(\xi,\eta)\mathrm{vol})\\
&= \langle\dot\xi,\eta\rangle_{\varphi(\mathbb{D})} + \langle\xi,\dot\eta\rangle_{\varphi(\mathbb{D})} + \langle\xi,\pounds_\xi\eta + 2\operatorname{div}(\xi)\eta\rangle_{\varphi(\mathbb{D})}
\end{aligned}
$$

and since the variation η vanishes at the endpoints, we get

$$0 = \langle\dot\xi + 2\operatorname{div}(\xi)\xi - \operatorname{grad}(|\xi|^2),\eta\rangle_{\varphi(\mathbb{D})} + \langle\xi,\pounds_\xi\eta\rangle_{\varphi(\mathbb{D})}.$$

Using now that $\pounds_\xi\eta = \xi'\eta - \eta'\xi = 2\xi'\eta - (\eta\xi)'$ we get

$$0 = \langle\dot\xi + 2\operatorname{div}(\xi)\xi - \operatorname{grad}(|\xi|^2) + 2\xi\overline{\xi'} - \bar\xi\partial_z^\top\xi,\eta\rangle_{\varphi(\mathbb{D})}.$$

Finally, from the relations $\operatorname{div}(\xi) = 2\operatorname{Re}(\xi')$ and $\operatorname{grad}(|\xi|^2) = 2\xi\overline{\xi'}$, and the decomposition in proposition 4.1, we obtain the strong geodesic equation

$$
\begin{aligned}
&\dot\xi + 2\operatorname{div}(\xi)\xi - \bar\xi\partial_z^\top\xi = \overline{\operatorname{grad}}(F) + \overline{\operatorname{sgrad}}(G)\\
&\dot\varphi = \xi\circ\varphi\\
&\frac{\partial\xi}{\partial\bar z} = 0\\
&F|_{\partial\varphi(\mathbb{D})} = G|_{\partial\varphi(\mathbb{D})} = 0.
\end{aligned}
\tag{9}
$$

Notice that the first equation contains the operator $\partial_z^\top$, which depends on the domain $\varphi(\mathbb{D})$. Thus, the first equation for $\dot\xi$ depends on the second equation for $\dot\varphi$. This is different from 'usual' Euler equations, where the equation for the reduced variable ξ is independent of φ. From a geometric mechanics point of view (cf [15]), the reason for this coupling is that the symmetry group of the Lagrangian is smaller than the configuration space. Also note that the EPDiff equations have no Lagrange multipliers corresponding to F, G, while the Eulerian fluid equations have one, the pressure, which can be eliminated by passing to the vorticity representation. It does not seem to be possible to eliminate the Lagrange multipliers F and G in (9).

For further information on this geodesic equation, its application in image registration, and an extension to H^1_α metrics, we refer the reader to [4].

Acknowledgments

This research was supported by the Swedish Research Council, the Massey University Research Fund and the Marsden Fund of the Royal Society of New Zealand. We thank

Tudor Ratiu for inspiration and advice on variational methods for embeddings, particularly during MP's visit to Caltech in 2010.

References

[1] Arnold V I 1966 *Ann. Inst. Fourier (Grenoble)* **16** 319–61
[2] Arnold V I and Khesin B A 1998 *Topological Methods in Hydrodynamics (Applied Mathematical Sciences* vol 125*)* (New York: Springer)
[3] Ebin D G and Marsden J E 1970 *Ann. Math.* **92** 102–63
[4] Marsland S, McLachlan R I, Modin K and Perlmutter M 2013 *Int. J. Comput. Vis.* **105** 144–54
[5] Holm D D and Marsden J E 2005 Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation *The Breadth of Symplectic and Poisson Geometry* (Berlin: Springer) pp 203–35
[6] Bruveris M, Gay-Balmaz F, Holm D D and Ratiu T S 2011 *J. Nonlinear Sci.* **21** 115–50
[7] Thompson D W 1917 *On Growth and Form* (Cambridge: Cambridge University Press)
[8] Hamilton R S 1982 *Bull. Am. Math. Soc.* **7** 65–222
[9] Abraham R, Marsden J E and Ratiu T 1988 *Manifolds, Tensor Analysis, and Applications (Applied Mathematical Sciences* vol 75*)* 2nd edn (New York: Springer)
[10] Schwarz G 1995 *Hodge Decomposition—a Method for Solving Boundary Value Problems (Lecture Notes in Mathematics* vol 1607*)* (Berlin: Springer)
[11] Cantarella J, DeTurck D and Gluck H 2002 *Amer. Math. Mon.* **109** 409–42
[12] Sharpe R W 1997 *Differential Geometry (Graduate Texts in Mathematics* vol 166*)* (New York: Springer)
[13] Duren P and Schuster A 2004 *Bergman Spaces (Mathematical Surveys and Monographs* vol 100*)* (Providence, RI: American Mathematical Society)
[14] Gay-Balmaz F, Marsden J E and Ratiu T S 2012 *J. Nonlinear Sci.* **22** 463–97
[15] Marsden J E and Ratiu T S 1999 *Introduction to Mechanics and Symmetry (Texts in Applied Mathematics* vol 17*)* 2nd edn (New York: Springer)