Summary of the Ph.D. thesis "Coupled Oscillator Networks" by Carlo Richard Laing

In this thesis I study the effects of forced symmetry breaking on the behaviour of a ring of three coupled modified van der Pol oscillators.

I start by examining the behaviour of the fully–symmetric system, which has $\mathbf{D}_3$ symmetry, using techniques from equivariant Hopf bifurcation theory to determine the types of orbit created in Hopf bifurcations. I then perform centre manifold and normal form calculations on the system studied to determine the stability of these orbits. I also use the theory of weakly coupled oscillators to show the existence of a number of orbits not created in Hopf bifurcations, and determine their stability. These two techniques predict different stability types for the branch of exactly synchronised oscillations. Interpolating between the parameter regimes where these analyses are valid I infer the existence of a $\mathbf{D}_3$ symmetric transcritical bifurcation of periodic orbits.

I then investigate the effects of breaking the $\mathbf{D}_3$ symmetry to $\mathbf{Z}_2$ and then removing it altogether on the Hopf bifurcations and the $\mathbf{D}_3$ symmetric transcritical bifurcation by examining the perturbed normal forms of these bifurcations. The study of the perturbed normal forms, as done by a number of authors, does not explain all the bifurcation behaviour, and details are completed numerically. Applications to the modelling of animal gaits is discussed. A partial investigation into the non–symmetric system is also undertaken which shows phase synchronisation, a new type of "switching bifurcation", and successive saddle–node bifurcations of n–tori on $(n+1)$–tori as a mechanism for the transition from motion on an N–torus to synchronised behaviour.

I also study the general case of N identical oscillators diffusively coupled in a ring, each of which when uncoupled undergo a supercritical Hopf bifurcation as a parameter μ is increased. I prove that for small enough coupling, the system as a whole undergoes a supercritical Hopf bifurcation to a stable rotating wave as μ is increased, and that the magnitude of the phase difference between neighbouring oscillators on this rotating wave is either a maximum or a minimum (i.e. zero), depending on the sign of the coupling constant.

Also included are a numerical study of bifocal homoclinic bifurcations in which some of the theoretical conjectures regarding these bifurcations are confirmed, and an introduction to the software package AUTO, the use of which is demonstrated on the Nosé equations.