143.223 Class test 1. Solutions. 2007

1. (10 marks) For the vectors $\mathbf{a} = (1\ \ 2)$ and $\mathbf{b} = (-1\ \ 0)$ find

   (i) $||\mathbf{a}||$ and $||\mathbf{b}||$

   $$||\mathbf{a}|| = \sqrt{1^2 + 2^2} = \sqrt{5}$$
   $$||\mathbf{b}|| = \sqrt{(-1)^2 + 0^2} = 1$$

   (ii) $\mathbf{a} \cdot \mathbf{b}$

   $$\mathbf{a} \cdot \mathbf{b} = (1\ \ 2) \cdot (-1\ \ 0) = -1 \times 1 + 0 \times 2 = -1$$

   (iii) $\mathbf{b} \cdot \mathbf{a}$

   $$\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = -1$$

   (iv) a unit vector $\mathbf{x}$ which satisfies $\mathbf{a} \cdot \mathbf{x} = 0$
   If $\mathbf{x} = (x\ \ y)$ then we have $x + 2y = 0 \Rightarrow x = -2y$. But $\mathbf{x}$ is a unit vector

   $$\Rightarrow x^2 + y^2 = 1 \Rightarrow (-2y)^2 + y^2 = 1 \Rightarrow 5y^2 = 1 \Rightarrow y = \pm 1/\sqrt{5}$$

   and $x = -2y = \mp 2/\sqrt{5}$. So there are two possible solutions, $\mathbf{x} = (\frac{-2}{\sqrt{5}}\ \ \frac{1}{\sqrt{5}})$ and $\mathbf{x} = (\frac{2}{\sqrt{5}}\ \ \frac{-1}{\sqrt{5}})$

2. (5 marks) If

   $$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

   calculate $A\mathbf{b}$ and $\mathbf{b}^T A$.

   $$A\mathbf{b} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

   $$\mathbf{b}^T A = (3\ \ 3) \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = (9\ \ 0)$$

3. (5 marks) Find the determinant of A, where

   $$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

   Expand down the first column:

   $$|A| = 1 \times \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} + 0 + 0 = 5 \times 1 - 2 \times 2 = 1$$

4. (10 marks) Show that for all real numbers, a, b and c, the eigenvalues of the matrix

   $$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

   are real.

   The eigenvalues λ satisfy

   $$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a - \lambda & c \\ c & b - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

$$\Rightarrow \lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab-c^2)}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4c^2}}{2}$$

The term under the square root is the sum of squares, so cannot be negative. Therefore, the eiegnvalues must be real.

5. (10 marks) Solve the simultaneous equations

$$\begin{aligned} 2x + 3y &= 1 \\ x + y &= -1 \end{aligned}$$

by writing them as a single matrix/vector equation and then finding the inverse of the matrix.

These can be writen as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The solution is $\mathbf{x} = A^{-1}\mathbf{b}$. Now

$$A^{-1} = \frac{1}{2 \times 1 - 3 \times 1} \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$

So

$$\mathbf{x} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $x = -4, y = 3$.

6. (10 marks) Write the vector $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ as a linear combination of the three vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

We need to find a, b, c such that

$$a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

Converting to simultaneous equations and forming the augmented matrix we obtain

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Gaussian elimination converts this to

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

i.e. $(a, b, c) = (1, -1, 2)$. Thus

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$