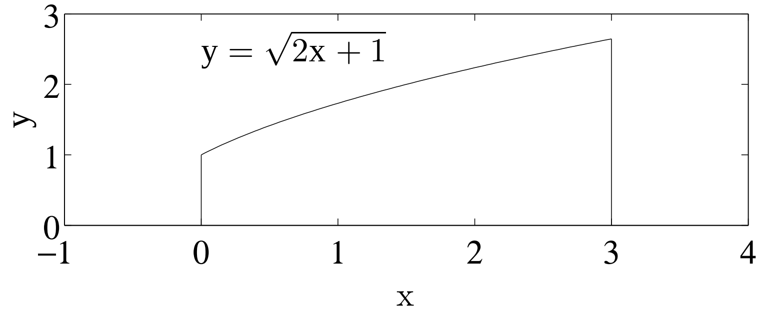
160.101 Class test 2 solutions, 2007

1. (5+5 marks)



(a) Find the area under the curve $y = \sqrt{2x+1}$ between for $0 \leq x \leq 3$, as shown in the figure.

$$\text{Area} = \int_0^3 \sqrt{2x+1}\,dx$$

Let $u = 2x+1$.

$$\Rightarrow \text{Area} = \int_1^7 \sqrt{u}\,du/2 = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^7 = (7^{3/2}-1)/3 \approx 5.8401$$

(b) Find the volume of the solid formed by rotating the area above one revolution about the x-axis.

$$\text{Volume} = \int_0^3 \pi(\sqrt{2x+1})^2 dx = \pi\int_0^3 2x+1\,dx = \pi\left[x^2+x\right]_0^3 = \pi(3^2+3) = 12\pi \approx 37.699$$

2. (6 marks) Use logarithmic differentiation to find the derivative of

$$y = \frac{2(x^2+1)}{\sqrt{\cos x}}$$

Write your answer in terms of x only.

$$\ln y = \ln\left(\frac{2(x^2+1)}{\sqrt{\cos x}}\right) = \ln 2 + \ln(x^2+1) - (1/2)\ln(\cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{\sin x}{2\cos x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{\tan x}{2}\right) = \frac{2(x^2+1)}{\sqrt{\cos x}}\left(\frac{2x}{x^2+1} + \frac{\tan x}{2}\right) = \frac{4x + (x^2+1)\tan x}{\sqrt{\cos x}}$$

3. (4+4 marks) Solve the following two equations for y:

(a) $3^y = 2^{y+1}$

$$y \log 3 = (y+1)\log 2 = y\log 2 + \log 2$$
$$\Rightarrow y(\log 3 - \log 2) = \log 2 \Rightarrow y = \frac{\log 2}{\log 3 - \log 2} = \frac{\log 2}{\log(3/2)} \approx 1.7095$$

Note: the base of the log doesn't matter.

(b) $\ln(y-1) = x + \ln(y)$

$$y - 1 = \exp(x + \ln y) = \exp x \exp(\ln y) = y \exp x$$
$$\Rightarrow y(1 - \exp x) = 1 \Rightarrow y = \frac{1}{1 - \exp x}$$

4. (10 marks) Find the following integral:

$$\int_3^4 \frac{x}{x^2 - 3x + 2} dx$$

By partial fractions

$$\frac{x}{x^2 - 3x + 2} = \frac{2}{x-2} - \frac{1}{x-1}$$
$$\Rightarrow \int_3^4 \frac{x}{x^2 - 3x + 2} dx = 2\int_3^4 \frac{dx}{x-2} - \int_3^4 \frac{dx}{x-1} = [2\ln(x-2) - \ln(x-1)]_3^4$$
$$= [2\ln 2 - \ln 3 - (2\ln 1 - \ln 2)] = 3\ln 2 - \ln 3 = \ln(8/3) \approx 0.98083$$

5. (8 marks) Find the following integral:

$$\int \frac{dt}{\sqrt{9 - 4t^2}}$$

Some useful formulas are at the bottom of the page. Write

$$\frac{1}{\sqrt{9 - 4t^2}} = \frac{1}{\sqrt{4(9/4 - t^2)}} = \frac{1}{2\sqrt{(3/2)^2 - t^2}}$$

Using the third formula

$$\int \frac{dt}{\sqrt{9 - 4t^2}} = \frac{1}{2}\sin^{-1}\left(\frac{t}{3/2}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2t}{3}\right) + C$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$$