Dynamics and Bifurcations of Piecewise-Smooth ODEs

Lecture Notes for the Summer Program on Dynamics of Complex Systems, ICTS

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Piecewise-smooth systems of ODEs are used in many areas to model physical phenomena involving impacts, switches, thresholds, and other discontinuous events. Such systems can have remarkably rich dynamics and exhibit transitions not possible in smooth systems (such as the local and instantaneous transition from equilibrium to chaos). Often the dynamics is captured by piecewise-linear equations, in which case key calculations may be tractable. These notes combine techniques from differential equations, analysis, and linear algebra to provide a quantitative introduction to piecewise-smooth dynamics.

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1 Linear systems

- A system of linear ODEs can be written as

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1.1}$$

where $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$, A is a real-valued $n \times n$ matrix, and the dot denotes differentiation with respect to time, t.

- If $Av = \lambda v$, then

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t} v, \tag{1.2}$$

is a solution to (1.1).

- Since (1.1) is linear, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to (1.1), then

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t), \tag{1.3}$$

is also a solution to (1.1).

- The idea is that the general solution to (1.1) can be written as a linear combination involving all eigenvalues and eigenvectors of A. Extra care is needed to deal with complex eigenvalues and repeated eigenvalues (for which we can use 'generalised eigenvectors' [1]). It turns out that the general solution can always be written as

$$\mathbf{x}(t) = \mathrm{e}^{tA}\mathbf{x}_0\,, \tag{1.4}$$

where matrix exponentiation is defined by

$$\mathrm{e}^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \tag{1.5}$$

Definition 1.1. Consider (1.1) (for which $\mathbf{0}$ is an equilibrium). Let $v_1, \ldots, v_n$ be a set of linearly independent generalised eigenvectors of A. For each j, write

$$v_j = u_j + \mathrm{i}w_j\,, \tag{1.6}$$

and let λ_j be the corresponding eigenvalue. The *stable subspace* of $\mathbf{0}$, denoted $E^s(\mathbf{0})$, is the span of all u_j and w_j for which $\mathrm{Re}(\lambda_j) < 0$. The *unstable subspace* of $\mathbf{0}$, denoted $E^s(\mathbf{0})$, is the span of all u_j and w_j for which $\mathrm{Re}(\lambda_j) > 0$.

- Now consider (1.1) in two dimensions. The eigenvalues of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1.7}$$

are

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}\,, \tag{1.8}$$

where

$$\tau = a + d\,, \qquad \delta = ad - bc\,, \tag{1.9}$$

are the trace and determinant of A.

- We can classify $\mathbf{0}$ in the following way:

  i) If $\tau > 0$ and $\delta < \frac{\tau^2}{4}$, then $0 < \lambda_- < \lambda_+$ and we refer to $\mathbf{0}$ as an *unstable node*.

ii) If $\tau < 0$ and $\delta < \frac{\tau^2}{4}$, then $\lambda_- < \lambda_+ < 0$ and we refer to $\mathbf{0}$ as a *stable node*.

iii) If $\delta < 0$, then $\lambda_- < 0 < \lambda_+$ and we refer to $\mathbf{0}$ as a *saddle*.

iv) If $\tau > 0$ and $\delta > \frac{\tau^2}{4}$, then λ_+ and λ_- are complex-valued with positive real parts and we refer to $\mathbf{0}$ as an *unstable focus*.

v) If $\tau < 0$ and $\delta > \frac{\tau^2}{4}$, then λ_+ and λ_- are complex-valued with negative real parts and we refer to $\mathbf{0}$ as a *stable focus*.

Exercise 1.1. Compute $E^s(\mathbf{0})$ and $E^u(\mathbf{0})$ for (1.1) with

$$A = \begin{bmatrix} 4 & -1 & -3 \\ 2 & 1 & -1 \\ 6 & -1 & -5 \end{bmatrix}.$$

Exercise 1.2. For the matrix

$$A = \begin{bmatrix} \alpha & -9 \\ 1 & 1 \end{bmatrix},$$

determine all $\alpha \in \mathbb{R}$ that correspond to each of the five cases listed above.

2 Equilibria of nonlinear systems

Definition 2.1 (Big-O notation)**.** A function $g(\mathbf{x})$ is said to be $\mathcal{O}(k)$ as $\mathbf{x} \to \tilde{\mathbf{x}}$ if

$$\limsup_{\mathbf{x} \to \tilde{\mathbf{x}}} \frac{\|g(\mathbf{x})\|}{\|\mathbf{x} - \tilde{\mathbf{x}}\|^k} < \infty. \tag{2.1}$$

- Let $\mathbf{x}^*$ be an equilibrium of an n-dimensional ODE system
$$\dot{\mathbf{x}} = f(\mathbf{x}). \tag{2.2}$$
If f is C^2 (has continuous second derivatives), then we can write the Taylor expansion of f about $\mathbf{x}^*$ as
$$f(\mathbf{x}) = Df(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + \mathcal{O}(2), \tag{2.3}$$
where $Df(\mathbf{x}^*)$ is the Jacobian matrix of f evaluated at $\mathbf{x}^*$ and the error terms are quadratic or higher order in $\mathbf{x} - \mathbf{x}^*$.

Example 2.1. Consider the system

$$\dot{x} = 4x^2 - y^2,$$
$$\dot{y} = x - y + 1.$$

Expanding about the equilibrium $(x, y) = (1, 2)$ produces

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} + \begin{bmatrix} 4(x-1)^2 - (y-2)^2 \\ 0 \end{bmatrix},$$

which has the form (2.3).

Definition 2.2. The *linearisation* of (2.2) about an equilibrium $\mathbf{x}^*$ is the linear system

$$\dot{\mathbf{x}} = Df(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*). \tag{2.4}$$

- Under the change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, (2.4) takes the familiar form $\dot{\mathbf{y}} = A\mathbf{y}$, where $A = Df(\mathbf{x}^*)$.
- Ostensibly, the nonlinear dynamics of (2.2) is approximated by the linear system (2.4), which we can understand completely. The Hartman-Grobman theorem gives a simple condition under which this approximation is 'valid' near $\mathbf{x}^*$.

Definition 2.3. An equilibrium $\mathbf{x}^*$ of (2.2) is said to be *hyperbolic* if $Df(\mathbf{x}^*)$ has no eigenvalues with zero real part.

Theorem 2.1 (Hartman-Grobman)**.** *Let $\mathbf{x}^*$ be a hyperbolic equilibrium of* (2.2)*, where f is C^1. Then there exists a neighbourhood of $\mathbf{x}^*$ within which* (2.2) *is conjugate*[1] *to its linearisation* (2.4).

Corollary 2.2. *If x^* is an equilibrium of* (2.2) *and all eigenvalues of $Df(\mathbf{x}^*)$ have negative real part, then x^* is asymptotically stable.*

- Further aspects of nonlinear ODEs are described in textbooks such as [1, 2, 3, 4].

Exercise 2.1. Consider the system

$$\dot{x} = x - y^2,$$
$$\dot{y} = (x - 2)(x - y - 2).$$

a) Find all equilibria.

[1] A *homeomorphism* is a function that is one-to-one, onto, continuous, and has a continuous inverse. Two systems of ODEs are said to be *conjugate* if there exists a homeomorphism that maps orbits of one system to orbits of the other system [1].

b) Classify each equilibrium as either (i) an unstable node, (ii) a stable node, (iii) a saddle, (iv) an unstable focus, (v) a stable focus, or (vi) other (e.g. non-hyperbolic).

c) Draw an accurate phase portrait: (i) plot the nullclines (curves where $\dot{x} = 0$ or $\dot{y} = 0$), (ii) indicate the equilibria, and (iii) sketch several typical orbits (using the results of (b) to help).

3 Bifurcations

- Next consider an n-dimensional ODE system that depends on a parameter $\mu \in \mathbb{R}$:

$$\dot{\mathbf{x}} = f(\mathbf{x}; \mu). \tag{3.1}$$

- Consider how the phase portrait of (3.1) varies as we change the value of μ by a small amount. Usually equilibria move a bit and their eigenvalues change a bit, but overall the dynamics remains essentially the same. Fundamental changes in the dynamics usually only occurs at critical values of μ, termed bifurcation values. Formally, bifurcations occur at values of μ for which (3.1) is not 'structurally stable' [1].
- A *bifurcation diagram* is a plot indicating invariant sets (e.g. equilibria) using a parameter on the horizontal axis and a variable on the vertical axis. Traditionally, stable and unstable sets are indicated with solid and dashed lines respectively (or blue and red lines, respectively, if colour is available).

Example 3.1. Consider the one-dimensional system

$$\dot{x} = x - x^3 + \mu. \tag{3.2}$$

Equilibria satisfy $0 = x - x^3 + \mu$. Also $Df(x) = 1 - 3x^2$, so equilibria with $|x| < \frac{1}{\sqrt{3}}$ are unstable and equilibria with $|x| > \frac{1}{\sqrt{3}}$ are stable. With this information we can plot the bifurcation diagram shown in Fig. 1.

As we increase the value of μ, the number of equilibria changes from 1 to 3 at $\mu = \frac{-2}{3\sqrt{3}}$, and then from 3 back to 1 at $\mu = \frac{2}{3\sqrt{3}}$. Hence (3.2) has bifurcations at $\mu = \frac{\pm 2}{3\sqrt{3}}$ (these are saddle-node bifurcations).

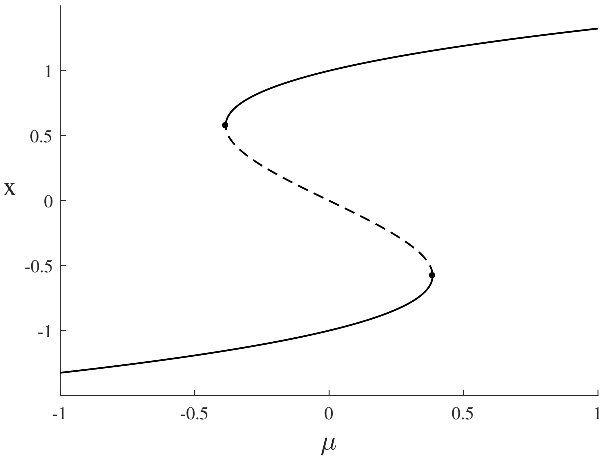

Figure 1: A bifurcation diagram of (3.2).

- Roughly speaking, a *saddle-node bifurcation* occurs when two equilibria collide and annihilate at some $\mu = \tilde{\mu}$. At $\tilde{\mu}$ the single equilibrium has an eigenvalue $\lambda = 0$. The distance of the equilibria from one another scales like $\sqrt{|\mu - \tilde{\mu}|}$.
- Roughly speaking, a *Hopf bifurcation* occurs when a limit cycle (isolated periodic orbit) emanates from an equilibrium at some $\mu = \tilde{\mu}$. At $\tilde{\mu}$ the equilibrium has eigenvalues $\lambda = \pm \mathrm{i}\omega$, for some $\omega > 0$. The amplitude of the limit cycle scales like $\sqrt{|\mu - \tilde{\mu}|}$, and its period tends to $\frac{2\pi}{\omega}$ as $\mu \to \tilde{\mu}$.
- For more details regarding classical bifurcations refer to the excellent book [5].

Exercise 3.1. Find all saddle-node bifurcations of

$$\begin{aligned} \dot{x} &= x - y, \\ \dot{y} &= x^2\left(1 - 3x + y^2\right) + \mu. \end{aligned} \tag{3.3}$$

4 A first look at Filippov systems: switching manifolds and sliding motion

- Our interest is with n-dimensional piecewise-smooth ODE systems of the form

$$\dot{\mathbf{x}} = \begin{cases} f_1(\mathbf{x}), & \mathbf{x} \in \Omega_1\,, \\ \quad\vdots & \\ f_m(\mathbf{x}), & \mathbf{x} \in \Omega_m\,. \end{cases} \tag{4.1}$$

Here phase space is partitioned into regions $\Omega_1, \ldots, \Omega_m \subset \mathbb{R}^n$. Within each Ω_i, orbits are governed by the smooth system $\dot{\mathbf{x}} = f_i(\mathbf{x})$. The boundaries of the Ω_i are assumed to be smooth $(n-1)$-dimensional manifolds, termed *switching manifolds*.

- *Filippov*[2] *systems* are piecewise-smooth systems with an additional rule (defined below) that governs 'sliding motion' on switching manifolds. Here we briefly introduce Filippov systems. For more details refer to [6] or Filippov's book [7].
- For the most part, the phenomena described below only involves one switching manifold, and it suffices to consider systems of the form

$$\dot{\mathbf{x}} = \begin{cases} f_L(\mathbf{x}), & H(\mathbf{x}) < 0, \\ f_R(\mathbf{x}), & H(\mathbf{x}) > 0. \end{cases} \tag{4.2}$$

- The equation $H(\mathbf{x}) = 0$ is assumed to define an $(n-1)$-dimensional surface: the switching manifold, call it Σ. The simplest way to ensure this is to assume that the gradient vector, $\nabla H(\mathbf{x})$, is a nonzero vector everywhere. Then, at each point, Σ is normal to $\nabla H(\mathbf{x})$.
- Let

$$\begin{aligned} v_L(\mathbf{x}) &= (\nabla H(\mathbf{x}))^{\mathsf{T}} f_L(\mathbf{x}), \\ v_R(\mathbf{x}) &= (\nabla H(\mathbf{x}))^{\mathsf{T}} f_R(\mathbf{x}). \end{aligned} \tag{4.3}$$

These represent the components of $f_L(\mathbf{x})$ and $f_R(\mathbf{x})$ in the direction $\nabla H(\mathbf{x})$. More precisely, if $\varphi_t^Z(\mathbf{x})$ denotes the *flow* ('solution to') $\dot{\mathbf{x}} = f_Z(\mathbf{x})$, where $Z \in \{L, R\}$, then[3]

$$\frac{d}{dt} H\big(\varphi_t^Z(\mathbf{x})\big)\Big|_{t=0} = v_Z(\mathbf{x}). \tag{4.4}$$

Definition 4.1. A subset of Σ is called

i) a *crossing region* if $v_L(\mathbf{x}) v_R(\mathbf{x}) > 0$,

ii) an *attracting sliding region* if $v_L(\mathbf{x}) > 0$ and $v_R(\mathbf{x}) < 0$, and

iii) a *repelling sliding region* if $v_L(\mathbf{x}) < 0$ and $v_R(\mathbf{x}) > 0$.

- The boundaries of these regions are $(n-2)$-dimensional surfaces, termed *fold surfaces*, where $v_L(\mathbf{x}) = 0$ $[v_R(\mathbf{x}) = 0]$. Through any point on such a surface, the orbit of f_L $[f_R]$ generically has a quadratic tangency with Σ. In this case if locally the orbit lies in $H(\mathbf{x}) \le 0$ $[H(\mathbf{x}) \ge 0]$ then the fold point is said to be *visible*, otherwise locally the orbit lies in $H(\mathbf{x}) \ge 0$ $[H(\mathbf{x}) \le 0]$ and the fold point is said to be *invisible*[4].
- When an orbit of (4.2) reaches a crossing region, we can assume it simply passes from one side of Σ to the other. However, when an orbit of (4.2) reaches an attracting sliding region, the given equations do not specify its subsequent motion in a classical sense.
- To remedy this, let us take an applied approach. Physical systems that rapidly switch between two states often involve hysteresis or time-delay to regulate the switching process (such as relay control systems in power electronics). A hysteretic system may be modelled by

$$\dot{\mathbf{x}} = \begin{cases} f_L(\mathbf{x}), & \text{until } H(\mathbf{x}) = \varepsilon, \\ f_R(\mathbf{x}), & \text{until } H(\mathbf{x}) = -\varepsilon, \end{cases} \tag{4.5}$$

where $\varepsilon > 0$ is small.

- An orbit of (4.5) is governed by f_R until $H(\mathbf{x}) = -\varepsilon$ at some point $\mathbf{z}$. Assuming $v_L(\mathbf{z}) > 0$, the orbit then reaches $H(\mathbf{x}) = \varepsilon$ after a time

$$t_L(\mathbf{z}) = \frac{2\varepsilon}{v_L(\mathbf{z})} + \mathcal{O}\big(\varepsilon^2\big), \tag{4.6}$$

and at a point $\mathbf{z} + t_L f_L(\mathbf{z}) + \mathcal{O}\big(\varepsilon^2\big)$. Assuming $v_R(\mathbf{z}) < 0$, the orbit then returns to $H(\mathbf{x}) = -\varepsilon$ after an additional time

$$t_R(\mathbf{z}) = \frac{-2\varepsilon}{v_R(\mathbf{z})} + \mathcal{O}\big(\varepsilon^2\big), \tag{4.7}$$

and at a point $\mathbf{z} + t_L f_L(\mathbf{z}) + t_R f_R(\mathbf{z}) + \mathcal{O}\big(\varepsilon^2\big)$.

- In summary, in a time $t_L + t_R$, the orbit has undergone two switches and been displaced by $t_L f_L + t_R f_R + \mathcal{O}\big(\varepsilon^2\big)$. To do $2k$ switches, the time taken is $k(t_L + t_R) + \mathcal{O}\big(\varepsilon^2\big)$, and the displacement is $k(t_L f_L + t_R f_R) + \mathcal{O}\big(\varepsilon^2\big)$, assuming $k\varepsilon$ is small. By taking $\varepsilon \to 0$ and $k \to \infty$ with $k\varepsilon \to 0$, we find that the orbit travels along Σ with an effective velocity of $\frac{k(t_L f_L + t_R f_R)}{k(t_L + t_R)}$, which simplifies to

$$f_S = \frac{v_L f_R - v_R f_L}{v_L - v_R}. \tag{4.8}$$

[2]Named after Aleksei Filippov (1932–2006) who developed a formal mathematical theory for such systems in the 1950's.
[3]This is the *Lie derivative* $\mathcal{L}_{F_Z} H$.
[4]Algebraically, visibility is governed by the sign of the second Lie derivative $\mathcal{L}_{F_Z}^2 H = \frac{d^2}{dt^2} H\big(\varphi_t^Z(\mathbf{x})\big)\big|_{t=0}$, see §7.2 of [6].

- Importantly, (4.8) arises if we instead use time-delay, certain numerical discretisations (e.g. forward Euler), a smooth approximation to (4.2) (under certain conditions), or add a small random component to (4.2)[5]. For this reason, when an orbit of (4.2) reaches an attracting sliding region, we will say that it subsequently evolves on Σ according to

$$\dot{\mathbf{x}} = f_S(\mathbf{x}). \tag{4.9}$$

  Such motion is called *sliding motion* and f_S is the *sliding vector field*.

- As written, (4.9) is n-dimensional, but really it is an $(n-1)$-dimensional system on Σ, and also applies to repelling sliding regions.

- Notice (4.8) is a convex combination of f_L and f_R. That is, $f_S = (1-\lambda)f_L + \lambda f_R$, for some $\lambda \in (0,1)$. Moreover, the value of λ is exactly that for which f_S is tangent to Σ.

- Sometimes f_S can be awkward to deal with because (4.8) is a quotient. Often we are only concerned with the paths that orbits take, not evolution times, in which case we may use the *scaled sliding vector field*

$$\hat{f}_S = v_L f_R - v_R f_L\,, \tag{4.10}$$

  assuming the sliding region is attracting.

Example 4.1. Here we consider an oscillator subject to stick-slip friction from a rotating belt [11, 12]. This is shown in Fig. 2 and modelled by the non-dimensionalised equations

$$\ddot{u} + b\dot{u} + u = -F_s\,\mathrm{sgn}(\dot{u} - v_0) + \kappa(\dot{u} - v_0). \tag{4.11}$$

Here v_0 is the speed of the belt, b is the damping coefficient, F_s is the maximum static friction, and $\kappa > 0$ is a constant used to approximate the Stribeck curve [13]. With $v = \dot{u}$, we can rewrite (4.11) as

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{cases} \begin{bmatrix} v \\ -u + (\kappa - b)v + F_s - \kappa v_0 \end{bmatrix}, & v < v_0, \\ \begin{bmatrix} v \\ -u + (\kappa - b)v - F_s - \kappa v_0 \end{bmatrix}, & v > v_0. \end{cases} \tag{4.12}$$

which is of the form (4.2). Fig. 3 shows a typical phase portrait.

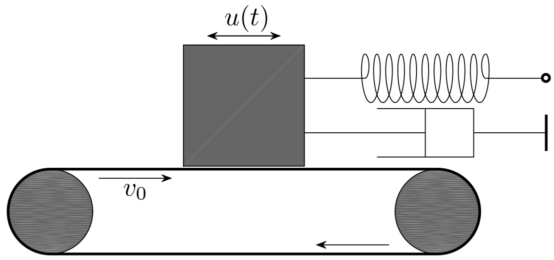

Figure 2: A schematic of the stick-slip friction oscillator modelled by (4.11).

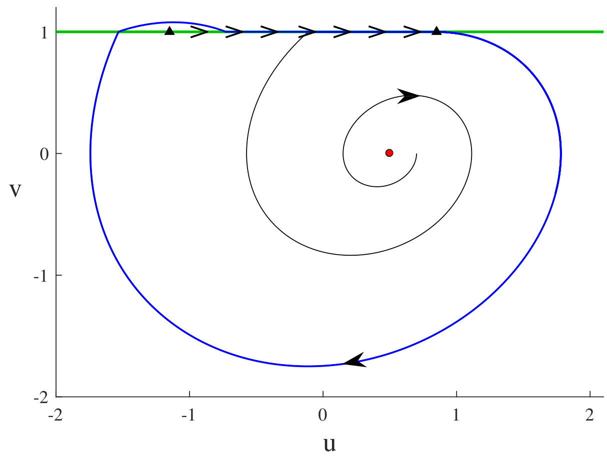

Figure 3: A phase portrait of (4.12) with $(v_0, b, F_s, \kappa) = (1, 0.15, 1, 0.5)$. The green line is the switching manifold $v = v_0$, and is an attracting sliding region between the fold points (indicated with triangles). The red dot is an unstable regular equilibrium; the blue loop is a stable limit cycle.

Exercise 4.1. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} y \\ 1 \end{bmatrix}, & x < 0, \\ \begin{bmatrix} -2(y-1) \\ -1 \end{bmatrix}, & x > 0, \end{cases} \tag{4.13}$$

a) Find the folds and classify them as visible or invisible.

[5] Many of these cases are covered by a remarkable theorem of Filippov (see Theorem 1 on pages 87–88 of [7]). Smoothing is achieved by 'Sotomayor-Teixeira regularisation' [8]. For randomness see [9, 10].

b) Classify subsets of the switching manifold as crossing regions, attracting sliding regions, and repelling sliding regions.

c) Compute the sliding vector field on the attracting sliding region.

d) Use the above results to sketch a phase portrait of the system.

Exercise 4.2. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & y < mx, \\ \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & y > mx, \end{cases} \tag{4.14}$$

where $a, m \in \mathbb{R}$.

a) Show that if $1 < m < 2$, then the switching manifold has an attracting sliding region. Hint: write $H(x,y) = y - mx$ and compute v_L and v_R.

b) Show that sliding motion on this region is directed towards the origin if and only if

$$a < \frac{m+3}{(m-1)(2-m)}. \tag{4.15}$$

Exercise 4.3. Consider the model of [14] for on/off PD control of an inverted pendulum

$$\begin{aligned} \dot{\theta} &= \phi, \\ \dot{\phi} &= \begin{cases} A\theta, & |\theta - \beta\phi| < \theta^*, \\ (A - K_p)\theta - K_d\phi, & |\theta - \beta\phi| > \theta^*, \end{cases} \end{aligned} \tag{4.16}$$

where $\theta(t)$ denotes the angle of the pendulum ($\theta = 0$ means the pendulum is vertical). Fix $A = \frac{1}{2}$, $K_p = 1$, $K_d = 1$, and $\theta^* = 1$, and focus on the behaviour of the system near $(\theta, \phi) = (1, 0)$.

a) Show that as the value of β is varied, two invisible folds collide at $\beta = 0$.

b) Extra: compute a Poincaré map to show that a stable limit cycle is created at $\beta = 0$ and has an amplitude asymptotically proportional to $\sqrt{\beta}$ (much like a Hopf bifurcation). Also give a physical interpretation of the limit cycle.

Exercise 4.4. Consider the Teixeira singularity (invisible-invisible two-fold) normal form [15]

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{cases} \begin{bmatrix} z \\ V^- \\ 1 \end{bmatrix}, & x < 0, \\ \begin{bmatrix} -y \\ 1 \\ V^+ \end{bmatrix}, & x > 0, \end{cases} \tag{4.17}$$

where $V^{\pm} \in \mathbb{R}$ are parameters.

a) Show that there are two curves of invisible folds that intersect at the origin.

b) Identify the attracting sliding region and derive the sliding vector field f_S.

c) Show that if

$$V^- < 0, \quad V^+ < 0, \quad V^- V^+ > 1, \tag{4.18}$$

then the scaled sliding vector field $\hat{f}_S$ is a linear system with a stable node at the origin (orbits actually arrive at the origin in finite time because f_S has a $\frac{0}{0}$ singularity).

d) Extra: compute a Poincaré map for one of the two crossing regions, assuming (4.18) holds. (The map is linear with trace $4V^-V^+ - 2$ and determinant 1.)

5 Non-Filippov behaviour and hidden dynamics

Example 5.1. Consider the following optimal control problem for the system

$$\dot{x} = ay, \qquad \dot{y} = b. \tag{5.1}$$

Given $x_0, y_0 > 0$, what control law $a = a(x,y)$, $b = b(x,y)$, subject to $|a|, |b| \le 1$, gets the forward orbit of (x_0, y_0) to the origin in the least possible time? The solution, given in [16], is:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, & x < \frac{y^2}{2}, \\ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, & x = \frac{y^2}{2}, \\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & x > \frac{y^2}{2}. \end{cases} \tag{5.2}$$

The system (5.1) with (5.2) is piecewise-smooth with switching manifold $x = \frac{y^2}{2}$. However, sliding motion is not governed by Filippov's convention (4.8), rather it is specified by the system itself.

- Dynamics near switching manifolds can be examined by smoothing or regularises the system in some way. This often introduces new dynamics that is otherwise 'hidden'.
- To smooth (4.2) we can write

$$\dot{\mathbf{x}} = \begin{cases} f_L(\mathbf{x}), & H(\mathbf{x}) \le -\varepsilon, \\ f_C(\mathbf{x}), & -\varepsilon \le H(\mathbf{x}) \le \varepsilon, \\ f_R(\mathbf{x}), & H(\mathbf{x}) \ge \varepsilon. \end{cases} \tag{5.3}$$

  where

$$f_C(\mathbf{x}) = \tfrac{1}{2}\left(1 - \tfrac{H(\mathbf{x})}{\varepsilon}\right) f_L(\mathbf{x}) + \tfrac{1}{2}\left(1 + \tfrac{H(\mathbf{x})}{\varepsilon}\right) f_R(\mathbf{x}) \\ - \left(1 - \left(\tfrac{H(\mathbf{x})}{\varepsilon}\right)^2\right) g\left(\tfrac{H(\mathbf{x})}{\varepsilon}\right), \tag{5.4}$$

  for some function g.
- With $g = \mathbf{0}$, the function f_C is a convex combination of f_L and f_R, and in the $\varepsilon \to 0$ limit (5.3) behaves like the Filippov system (4.2) (with sliding motion governed by f_S).
- For physical reasons we may choose $g \ne \mathbf{0}$ with which (5.3) may exhibit 'non-Filippov' dynamics, as illustrated in the next example. The function g is independent of f_L and f_R and poetically described by Jeffrey as the 'ghost of departed quantities' [17].

Example 5.2. Consider again the stick-slip friction oscillator of Fig. 2, but now modelled by

$$\ddot{u} + b\dot{u} + u = -F_s \phi(\dot{u}) + \kappa(\dot{u} - v_0), \tag{5.5}$$

where

$$\phi(v) = \begin{cases} \mathrm{sgn}(v - v_0), & |v - v_0| \ge \varepsilon, \\ \frac{v - v_0}{\varepsilon} + \gamma\left(\frac{v - v_0}{\varepsilon} - \left(\frac{v - v_0}{\varepsilon}\right)^3\right), & \text{o.w.}, \end{cases} \tag{5.6}$$

and $\gamma > 0$ is a constant. This particular smoothing further models the Stribeck effect and is of the form (5.3), see Exercise 5.1. Fig. 4 shows a typical phase portrait. Notice how the limit cycle remains constrained near the switching manifold for some distance past the right-most fold point (black triangle) into where the switching manifold is a crossing region [18].

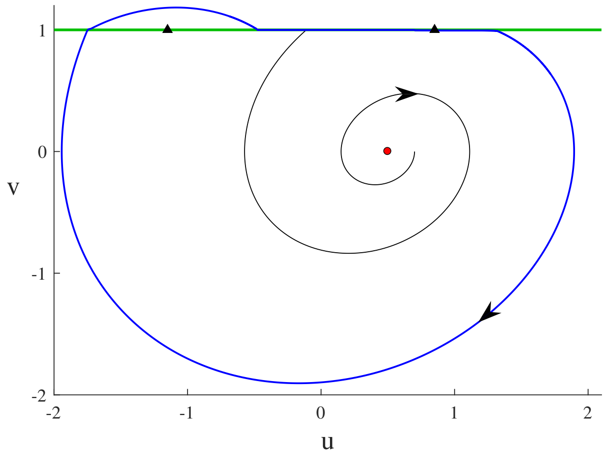

Figure 4: A phase portrait of the smoothed system (5.5) with $\varepsilon = 0.01$, $\gamma = 2$, and the same parameters as Fig. 3.

- The next example shows that at the intersection of two or more switching manifolds, non-Filippov dynamics can be generated even with linear smoothing (i.e. $g = \mathbf{0}$) [19].

Example 5.3. Consider the four-piece Filippov system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} f_1(x,y), & x > 0, \; y > 0, \\ f_2(x,y), & x < 0, \; y > 0, \\ f_3(x,y), & x < 0, \; y < 0, \\ f_4(x,y), & x > 0, \; y > 0, \end{cases} \tag{5.7}$$

and the smoothed version

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \tfrac{1}{4}\left(1 + \tfrac{x}{\varepsilon}\right)\left(1 + \tfrac{y}{\varepsilon}\right) f_1(x,y) + \tfrac{1}{4}\left(1 - \tfrac{x}{\varepsilon}\right)\left(1 + \tfrac{y}{\varepsilon}\right) f_2(x,y) \\ + \tfrac{1}{4}\left(1 - \tfrac{x}{\varepsilon}\right)\left(1 - \tfrac{y}{\varepsilon}\right) f_3(x,y) + \tfrac{1}{4}\left(1 + \tfrac{x}{\varepsilon}\right)\left(1 - \tfrac{y}{\varepsilon}\right) f_4(x,y). \tag{5.8}$$

With the constant vectors

$$f_1 = \begin{bmatrix} 1 \\ -0.4 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \tag{5.9}$$

orbits of (5.7) spiral outwards. Yet (5.8) has a stable equilibrium whose basin of attraction is bounded by an unstable limit cycle, Fig. 5.

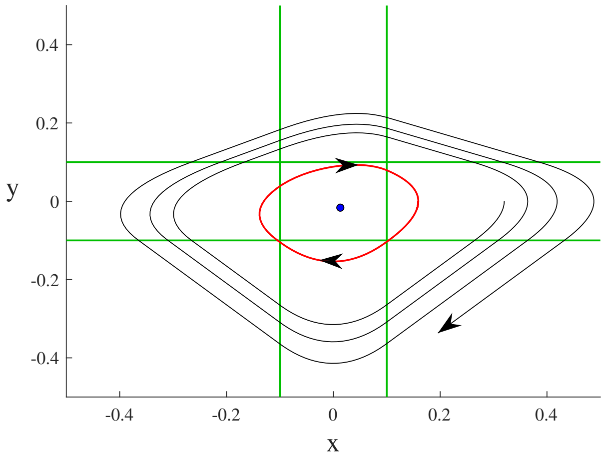

Figure 5: A phase portrait of the smoothed system (5.8) with $\varepsilon = 0.1$.

Exercise 5.1. Consider the stick-slip friction oscillator in the form (4.12). Show that with $H(u, v) = v - v_0$ and $g(\lambda) = \gamma F_s \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the smoothed system (5.3)–(5.4) can be rewritten as (5.5)–(5.6).

Exercise 5.2. Consider the Filippov system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \beta + \mathrm{sgn}(x) \\ -y \end{bmatrix}, \tag{5.10}$$

and the smoothed system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \beta + \psi\left(\frac{x}{\varepsilon}\right) \\ -y \end{bmatrix}, \tag{5.11}$$

where

$$\psi(\lambda) = \begin{cases} \mathrm{sgn}(\lambda), & |\lambda| \geq 1, \\ 2\lambda^2 + \lambda - 2, & |\lambda| \leq 1, \end{cases}$$

Show that if $1 < \beta < \frac{17}{8}$, then $x = 0$ is a crossing region of (5.10) yet orbits of (5.11) become trapped near $x = 0$ for arbitrarily small $\varepsilon > 0$.

6 Equilibria of Filippov systems

Definition 6.1. A *regular equilibrium* of (4.2) is a point $\mathbf{x}^* \in \mathbb{R}^n$ for which $f_L(\mathbf{x}^*) = \mathbf{0}$ $[f_R(\mathbf{x}^*) = \mathbf{0}]$, and said to be

i) *admissible* if $H(\mathbf{x}^*) < 0$ $[H(\mathbf{x}^*) > 0]$,

ii) *virtual* if $H(\mathbf{x}^*) > 0$ $[H(\mathbf{x}^*) < 0]$, and

iii) a *boundary equilibrium* if $H(\mathbf{x}^*) = 0$.

Definition 6.2. A *pseudo-equilibrium* of (4.2) is a point $\mathbf{x}^*$ on $H(\mathbf{x}) = 0$ for which $f_S(\mathbf{x}^*) = \mathbf{0}$, and said to be

i) *admissible* if $v_L(\mathbf{x}^*) v_R(\mathbf{x}^*) < 0$, and

ii) *virtual* if $v_L(\mathbf{x}^*) v_R(\mathbf{x}^*) > 0$.

- To determine the stability of a pseudo-equilibrium $\mathbf{x}^*$, we can reformulate $\dot{\mathbf{x}} = f_S(\mathbf{x})$ as an $(n-1)$-dimensional vector field and apply Corollary 2.2.
- If $\tilde{M}$ denotes the $(n-1) \times (n-1)$ Jacobian matrix of this vector field evaluated at $\mathbf{x}^*$, then the eigenvalues of $D f_S(\mathbf{x}^*)$ are 0 and those of $\tilde{M}$, see Exercise 6.4.

Example 6.1. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, & 2x - y < 0, \\ \begin{bmatrix} -2 \\ x - y \end{bmatrix}, & 2x - y > 0. \end{cases} \tag{6.1}$$

Using $H(x, y) = 2x - y$ we obtain $v_L(x, y) = 5$ and $v_R(x, y) = x - 4$ (on $y = 2x$), and so $y = 2x$ is an attracting sliding region for $x < 4$. By evaluating (4.8) we find that on this region we have

$$\dot{x} = \frac{-3x + 2}{-x + 9}.$$

Solving $\dot{x} = 0$ gives the pseudo-equilibrium $(x, y) = \left(\frac{2}{3}, \frac{4}{3}\right)$. This equilibrium is stable because the sliding region is attracting and $\frac{d}{dx}\dot{x}\,\big|_{x=\frac{2}{3}} = -\frac{9}{25} < 0$.

Exercise 6.1. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} y \\ x + 1 \end{bmatrix}, & x < 0, \\ \begin{bmatrix} -1 \\ -2 \end{bmatrix}, & x > 0, \end{cases} \tag{6.2}$$

a) Compute equilibria, sliding motion, etc, and use these to sketch a phase portrait.

b) What bounds the basin of attraction of the stable equilibrium?

Exercise 6.2. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{cases} \begin{bmatrix} z+3 \\ x \\ y+1 \end{bmatrix}, & y < z, \\ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, & y > z. \end{cases} \tag{6.3}$$

a) Compute the pseudo-equilibrium $\mathbf{x}^*$.

b) Is $\mathbf{x}^*$ admissible, and if so does $\mathbf{x}^*$ belong to an attracting sliding region or a repelling sliding region?

c) In the context of the two-dimensional sliding vector field, classify $\mathbf{x}^*$ as in Exercise 2.1b.

Exercise 6.3. Consider (4.8) at a point $\mathbf{x}$ for which $f_L, f_R \neq \mathbf{0}$. Show that $f_S = \mathbf{0}$ if and only if f_L and f_R are linearly dependent (i.e. point in the same direction).

Exercise 6.4. Consider (4.8) and suppose $f_S(\mathbf{x}^*) = \mathbf{0}$. Show that 0 is an eigenvalue of $Df_S(\mathbf{x}^*)$ with left eigenvector $(\nabla H(\mathbf{x}^*))^{\mathsf{T}}$.

7 BEBs in Filippov systems

- A *boundary equilibrium bifurcation* (BEB) occurs when a regular equilibrium collides with a switching manifold as parameters are varied. A wide variety of invariant sets can be created in BEBs (such as limit cycles and chaotic attractors).

- We consider Filippov systems of the form

$$\dot{\mathbf{x}} = \begin{cases} f_L(\mathbf{x};\mu), & H(\mathbf{x}) < 0, \\ f_R(\mathbf{x};\mu), & H(\mathbf{x}) > 0, \end{cases} \tag{7.1}$$

for which the switching manifold, call it Σ, passes through the origin. Thus $H(\mathbf{0}) = 0$, and we assume $\nabla H(\mathbf{0}) \neq \mathbf{0}$. For brevity we let

$$\zeta = \nabla H(\mathbf{0}). \tag{7.2}$$

- Now suppose f_L has an equilibrium at $\mathbf{x} = \mathbf{0}$ when $\mu = 0$. Then we can write

$$f_L(\mathbf{x};\mu) = A\mathbf{x} + b\mu + \mathcal{O}(2), \tag{7.3}$$
$$f_R(\mathbf{x};\mu) = c + \mathcal{O}(1), \tag{7.4}$$

where A is an $n \times n$ matrix and $b, c \in \mathbb{R}^n$. In (7.3)–(7.4), $\mathcal{O}(k)$ represents terms that are order k or greater in $\mathbf{x}$ and μ.

- If $\det(A) \neq 0$ and $c \neq \mathbf{0}$, then, locally, (7.1) has the unique regular equilibrium

$$\mathbf{x}_L(\mu) = -A^{-1}b\mu + \mathcal{O}\big(\mu^2\big). \tag{7.5}$$

- By evaluating (4.8), we obtain

$$f_S(\mathbf{x};\mu) = \left(I - \frac{c\zeta^{\mathsf{T}}}{\zeta^{\mathsf{T}}c}\right)(A\mathbf{x} + b\mu) + \mathcal{O}(2), \tag{7.6}$$

assuming $v_R(\mathbf{0};0) = \zeta^{\mathsf{T}}c \neq 0$. Let

$$M = Df_S(\mathbf{0};0) = \left(I - \frac{c\zeta^{\mathsf{T}}}{\zeta^{\mathsf{T}}c}\right)A. \tag{7.7}$$

- At this point our n-dimensional formulation of f_S becomes a little inconvenient. Let $\{u_1, \ldots, u_{n-1}\}$ be a basis for the orthogonal complement of ζ (these vectors span the tangent plane of Σ at $\mathbf{x} = \mathbf{0}$), and let $U = \big[u_1 \ldots u_{n-1}\big]$. If we were to rewrite the sliding vector field as an $(n-1)$-dimensional system using $u_1, \ldots, u_{n-1}$ to define coordinates on Σ, then the Jacobian matrix of this system evaluated at $(\mathbf{x};\mu) = (\mathbf{0};0)$ would be

$$\tilde{M} = U^{\mathsf{T}}MU. \tag{7.8}$$

The eigenvalues of $\tilde{M}$ are independent of the particular basis vectors used.

- If $\det(\tilde{M}) \neq 0$ and $\zeta^{\mathsf{T}}c \neq 0$, then, locally, (7.1) has a unique regular pseudo-equilibrium, $\mathbf{x}_S(\mu)$. Also $\mathbf{x}_S(0) = \mathbf{0}$ (by uniqueness). The next result provides a simple formula for $\det(\tilde{M})$.

Lemma 7.1. *Suppose* $\det(A) \neq 0$ *and* $\zeta^{\mathsf{T}}c \neq 0$. *Then*

$$\det(\tilde{M}) = \frac{\varrho^{\mathsf{T}}c}{\zeta^{\mathsf{T}}c}, \tag{7.9}$$

where

$$\varrho^{\mathsf{T}} = \zeta^{\mathsf{T}}\mathrm{adj}(A), \tag{7.10}$$

and $\mathrm{adj}(A)$ *is the adjugate*[6] *of* A.

[6] Let m_{ij} be the determinant of the $(n-1)\times(n-1)$ matrix formed by removing the $i^{\rm th}$ row and $j^{\rm th}$ column from A (these are the *minors* of A). The *cofactor matrix* of A has elements $(-1)^{i+j}m_{ij}$. The *adjugate* of A is the transpose of the cofactor matrix and satisfies $\mathrm{adj}(A)A = \det(A)I$. Thus, if $\det(A) \neq 0$, then $A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$.

- The all important inner product $\varrho^{\mathsf{T}} c$ is identified in §5.2 of [6], but the connection of this to $\tilde{M}$ and to the Jacobian of the sliding vector field seems to be a new observation [20]!

Proof of Lemma 7.1. The characteristic polynomials of M and $\tilde{M}$ are related by

$$\det(\lambda I - M) = \lambda \det(\lambda I - \tilde{M}). \tag{7.11}$$

To evaluate $\det(\lambda I - M)$ we use the matrix determinant lemma:

$$\det(X + vu^{\mathsf{T}}) = \det(X)\Big(1 + u^{\mathsf{T}} X^{-1} v\Big), \tag{7.12}$$

for any non-singular $n \times n$ matrix X and $u, v \in \mathbb{R}^n$. From (7.7) we obtain

$$\det(\lambda I - M) = \det(\lambda I - A)\left(1 + \frac{\zeta^{\mathsf{T}} A(\lambda I - A)^{-1} c}{\zeta^{\mathsf{T}} c}\right).$$

By then substituting $(\lambda I - A)^{-1} = -A^{-1} - A^{-2}\lambda + \mathcal{O}\big(\lambda^2\big)$, we obtain

$$\det(\lambda I - M) = -\frac{\det(-A)}{\zeta^{\mathsf{T}} c}\, \zeta^{\mathsf{T}} A^{-1} c \lambda + \mathcal{O}\big(\lambda^2\big).$$

By (7.10) this reduces to

$$\det(\lambda I - M) = \frac{(-1)^{n+1} \varrho^{\mathsf{T}} c}{\zeta^{\mathsf{T}} c}\, \lambda + \mathcal{O}\big(\lambda^2\big).$$

Thus by (7.11) we have $\det(-\tilde{M}) = \frac{(-1)^{n+1} \varrho^{\mathsf{T}} c}{\zeta^{\mathsf{T}} c}$, and hence (7.9), as required. □

- Now let us think about how the admissibility of $\mathbf{x}_L$ and $\mathbf{x}_S$ change as the value of μ changes sign. Write

$$\begin{aligned} H(\mathbf{x}_L(\mu)) &= \alpha_L \mu + \mathcal{O}\big(\mu^2\big), \\ v_L(\mathbf{x}_S(\mu); \mu) v_R(\mathbf{x}_S(\mu); \mu) &= \alpha_S \mu + \mathcal{O}\big(\mu^2\big), \end{aligned}$$

for some $\alpha_L, \alpha_S \in \mathbb{R}$. Then $\mathbf{x}_L$ is admissible if $\alpha_L \mu < 0$, and $\mathbf{x}_S$ is admissible if $\alpha_S \mu < 0$. If $\mathbf{x}_L$ and $\mathbf{x}_S$ are admissible for different signs of μ [the same sign of μ], then the BEB is referred to as *persistence* [a *nonsmooth-fold*]. Immediately we have the following result.

Lemma 7.2. *The BEB at $\mu = 0$ corresponds to persistence if $\alpha_L \alpha_S < 0$ and to a nonsmooth-fold if $\alpha_L \alpha_S > 0$.*

- Next we give formulas for α_L and α_S.

Lemma 7.3. *Suppose* $\det(A) \neq 0$, $\det(\tilde{M}) \neq 0$, *and* $\zeta^{\mathsf{T}} c \neq 0$. *Then*

$$\alpha_L = -\frac{\varrho^{\mathsf{T}} b}{\det(A)}, \qquad \alpha_S = \frac{\varrho^{\mathsf{T}} b\, \zeta^{\mathsf{T}} c}{\det(\tilde{M})}. \tag{7.13}$$

- The proof of Lemma 7.3 is left as an exercise.
- In view of (7.13), the *transversality condition* for the BEB is $\varrho^{\mathsf{T}} b \neq 0$.
- By combining the above results we obtain the following theorem which indicates how the BEB can be classified in terms of the eigenvalues associated with $\mathbf{x}_L$ and $\mathbf{x}_S$. This type of result was first achieved by Feigin for piecewise-linear maps [21].

Theorem 7.4. *Suppose* $\det(A) \neq 0$, $\det(\tilde{M}) \neq 0$, $\varrho^{\mathsf{T}} b \neq 0$, *and* $\zeta^{\mathsf{T}} c \neq 0$. *Then*

$$\operatorname{sgn}(\alpha_L \alpha_S) = (-1)^{N_L + N_S} \operatorname{sgn}\Big(\zeta^{\mathsf{T}} c\Big), \tag{7.14}$$

where N_L is the number of real positive eigenvalues of A, and N_S is the number of real positive eigenvalues of $\tilde{M}$.

Proof. By Lemma 7.3,

$$\operatorname{sgn}(\alpha_L \alpha_S) = -\operatorname{sgn}\Big(\det(A) \det(\tilde{M})\, \zeta^{\mathsf{T}} c\Big). \tag{7.15}$$

The result then follows by observing: $(-1)^{N_L} = \operatorname{sgn}(\det(-A))$ and $(-1)^{N_S} = \operatorname{sgn}(\det(-\tilde{M}))$. □

Exercise 7.1. Prove Lemma 7.3. Hints: the formula for α_L is essentially trivial; v_R is given by $\zeta^{\mathsf{T}} c$, to leading order, so it remains to evaluate $v_L = \zeta^{\mathsf{T}} f_L$ at $\mathbf{x}_S$; this can be achieved by substituting $\mathbf{x}_S$ into

$$f_S(\mathbf{x}; \mu) = \left(I - \frac{c \zeta^{\mathsf{T}}}{\zeta^{\mathsf{T}} c}\right) f_L(\mathbf{x}; \mu) + \mathcal{O}(2), \tag{7.16}$$

and simplifying.

Exercise 7.2. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{cases} \begin{bmatrix} x + \mu \\ 2y + \mu \\ 3z + k\mu \end{bmatrix}, & x + y + z < 0, \\ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, & x + y + z > 0, \end{cases} \tag{7.17}$$

a) Show that the BEB at $\mu = 0$ satisfies the transversality condition, $\varrho^{\mathsf{T}} b \neq 0$, if and only if $k \neq -\frac{9}{2}$.

b) Evaluate M by using the formula (7.7), and compute its eigenvalues.

c) Use Theorem 7.4 to classify the BEB as either persistence or a nonsmooth-fold.

Exercise 7.3. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & x + y < \xi, \\ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, & x + y > \xi. \end{cases} \tag{7.18}$$

Use Theorem 7.4 to classify the BEB at $\xi = 0$ for different values of $a \in \mathbb{R}$.

8 The normal form for BEBs in Filippov systems

- Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. For any $\mathbf{x} \in \mathbb{R}^n$, let

$$x = e_1^{\mathsf{T}} \mathbf{x}, \tag{8.1}$$

denote its first component.

- Given a Filippov system of the form (7.1), we assume a coordinate change can be performed so that, at least locally, the switching function becomes simply $H(\mathbf{x}) = x$ (and so $\zeta = e_1$).
- As in the previous section, we assume f_L has an equilibrium at $\mathbf{x} = \mathbf{0}$ when $\mu = 0$. Thus the system has the form

$$\dot{\mathbf{x}} = \begin{cases} A\mathbf{x} + b\mu + \mathcal{O}(2), & x < 0, \\ c + \mathcal{O}(1), & x > 0, \end{cases} \tag{8.2}$$

- The *BEB normal form* for Filippov systems is

$$\dot{\mathbf{x}} = \begin{cases} C\mathbf{x} + e_n\mu, & x < 0, \\ d, & x > 0, \end{cases} \tag{8.3}$$

where C is a *companion matrix* of the form

$$C = \begin{bmatrix} -a_1 & 1 & & \\ -a_2 & & \ddots & \\ \vdots & & & 1 \\ -a_n & & & \end{bmatrix}, \tag{8.4}$$

and $d \in \mathbb{R}^n$ with $d_1 = \pm 1$, see [20].

- The normal form has $2n - 1$ parameters (not counting $d_1 = \pm 1$ or the BEB parameter μ). By scaling, it suffices to consider $\mu \in \{-1, 0, 1\}$.
- Conveniently the numbers $a_i \in \mathbb{R}$ are the coefficients of the characteristic polynomial of C:

$$\det(\lambda I - C) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \,. \tag{8.5}$$

- For a given system (8.2), we would like to transform it to the normal form (plus higher order terms) via an invertible coordinate change

$$\mathbf{x} \mapsto Q\mathbf{x} + r\mu, \qquad \mu \mapsto s\mu, \tag{8.6}$$

that does not disrupt the switching manifold $x = 0$. This requires (i) $e_1^{\mathsf{T}} Q = \kappa e_1^{\mathsf{T}}$, for some $\kappa > 0$, (ii) $e_1^{\mathsf{T}} r = 0$, (iii) $\det(Q) \neq 0$, (iv) $s \neq 0$. By adapting standard results from control theory [22, 23], it can be shown that such a coordinate change is possible if and only if (8.2) is 'observable' and $\varrho^{\mathsf{T}} b \neq 0$.

Definition 8.1. We say that (8.2) is *observable* if $\det(\Phi) \neq 0$, where

$$\Phi = \begin{bmatrix} e_1^{\mathsf{T}} \\ e_1^{\mathsf{T}} A \\ \vdots \\ e_1^{\mathsf{T}} A^{n-1} \end{bmatrix}. \tag{8.7}$$

- It can also be shown that (8.2) is observable if and only if A has no eigenvector orthogonal to e_1 (this is essentially the Popov-Belevitch-Hautus test [23, 24]).
- Next we explain how (8.6) can be constructed. Given a system of the form (8.2), let $a_1, \ldots, a_n$ be the coefficients of the characteristic polynomial of A (in the order matching (8.5)), and use these to form the matrix

$$\Psi = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}. \tag{8.8}$$

Let J be the companion matrix (8.4) for which $a_i = 0$ for all i. Then the desired coordinate change is

$$Q = \frac{1}{|e_1^{\mathsf{T}} c|} \Psi \Phi, \tag{8.9}$$

$$r = J^{\mathsf{T}} Q b, \tag{8.10}$$

$$s = e_n^{\mathsf{T}} Q b, \tag{8.11}$$

assuming $e_1^{\mathsf{T}} c \neq 0$, see Exercise 8.4.

- For the normal form (8.3), the sliding vector field is

$$f_S(\mathbf{x}; \mu) = \frac{M\mathbf{x} + e_n \mu}{1 - \frac{e_1^{\mathsf{T}}(C\mathbf{x} + e_n \mu)}{e_1^{\mathsf{T}} d}}, \tag{8.12}$$

where $M = \left(I - \frac{d e_1^{\mathsf{T}}}{e_1^{\mathsf{T}} d} \right) C$.

- By scaling (similar to $\hat{f}_S$ (4.10)) the denominator of (8.12) can be neglected. We can also reformulate (8.12) as an $(n-1)$-dimensional system by setting $x = 0$ and ignoring the first component $\dot{x}$. Writing $\mathbf{x} = (x, \mathbf{y})$ we arrive at

$$\dot{\mathbf{y}} = \tilde{M} \mathbf{y} + e_{n-1} \mu, \tag{8.13}$$

where

$$\tilde{M} = \begin{bmatrix} -\frac{d_2}{d_1} & 1 & & \\ -\frac{d_3}{d_1} & & \ddots & \\ \vdots & & & 1 \\ -\frac{d_n}{d_1} & & & \end{bmatrix}, \tag{8.14}$$

is the matrix obtained by removing the first row and column of M, and, with a slight abuse of notation, $e_{n-1} = [0, \ldots, 0, 1]^{\mathsf{T}} \in \mathbb{R}^{n-1}$.

- In two dimensions we can write the normal form (8.3) as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \mu \end{bmatrix}, & x < 0, \\ \begin{bmatrix} -1 \\ d_2 \end{bmatrix}, & x > 0, \end{cases} \tag{8.15}$$

in the case $d_1 = -1$. When $\mu = 0$ the origin is a boundary equilibrium of which there are eight topologically distinct generic types, see Fig. 6. These are 'unfolded' by (8.15). As indicated in the figure, some boundary equilibria have more than one unfolding (see also Exercise 8.6) and there are a total of 12 topologically distinct generic BEBs in two dimensions [25, 26].

- In three dimensions, (8.3) can exhibit chaos [27].

Exercise 8.1. Consider again the system (7.17).

a) Apply a linear coordinate change to put this system in the form (8.2) for some A, b, and c.

b) Compute C in the BEB normal form. Hint: you do not need to use the transformation (8.6), just evaluate the coefficients of the characteristic polynomial of A.

c) Evaluate Φ, Ψ, and Q, and then compute the vector d in the BEB normal form by evaluating $d = Qc$.

d) Here you will again compute d, but without explicitly using the transformation (8.6). First evaluate the eigenvalues of $\tilde{M}$ (done in Exercise 7.2). Use these values to determine d_2 and d_3 via (8.14), where $d_1 = \mathrm{sgn}(c_1)$. Make sure your answer is the same as what you obtained in (c).

Exercise 8.2. Show that $\det(\Phi) \neq 0$, where Φ is given by (8.7), for any 2×2 matrix A with complex (non-real) eigenvalues.

Exercise 8.3. Give an example of a matrix A (necessarily at least 4-dimensional) that has no real eigenvalues, yet $\det(\Phi) = 0$.

Exercise 8.4. Here you will verify (8.9)–(8.11). By omitting the higher order terms of (8.2) (for simplicity) and introducing new symbols (for clarity), we obtain

$$\dot{\mathbf{z}} = \begin{cases} A\mathbf{z} + b\nu, & z < 0, \\ c, & z > 0, \end{cases} \tag{8.16}$$

where $z = e_1^{\mathsf{T}} \mathbf{z}$.

a) Show that under the coordinate change

$$\mathbf{x} = Q\mathbf{z} + r\nu, \qquad \mu = s\nu, \tag{8.17}$$

the system (8.16) becomes

$$\dot{\mathbf{x}} = \begin{cases} QAQ^{-1}\mathbf{x} + \frac{1}{s}\left(Qb - QAQ^{-1} r \right) \mu, & x < 0, \\ Qc, & x > 0. \end{cases} \tag{8.18}$$

b) Use the Cayley-Hamilton theorem[7] to show that

$$J\Psi\Phi - \Psi\Phi A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} e_1^{\mathsf{T}}. \tag{8.19}$$

[7] Every square matrix satisfies its own characteristic equation. That is, if $p(\lambda) = \det(\lambda I - A)$ denotes the characteristic polynomial of A, then $p(A) = O$ (the zero matrix).

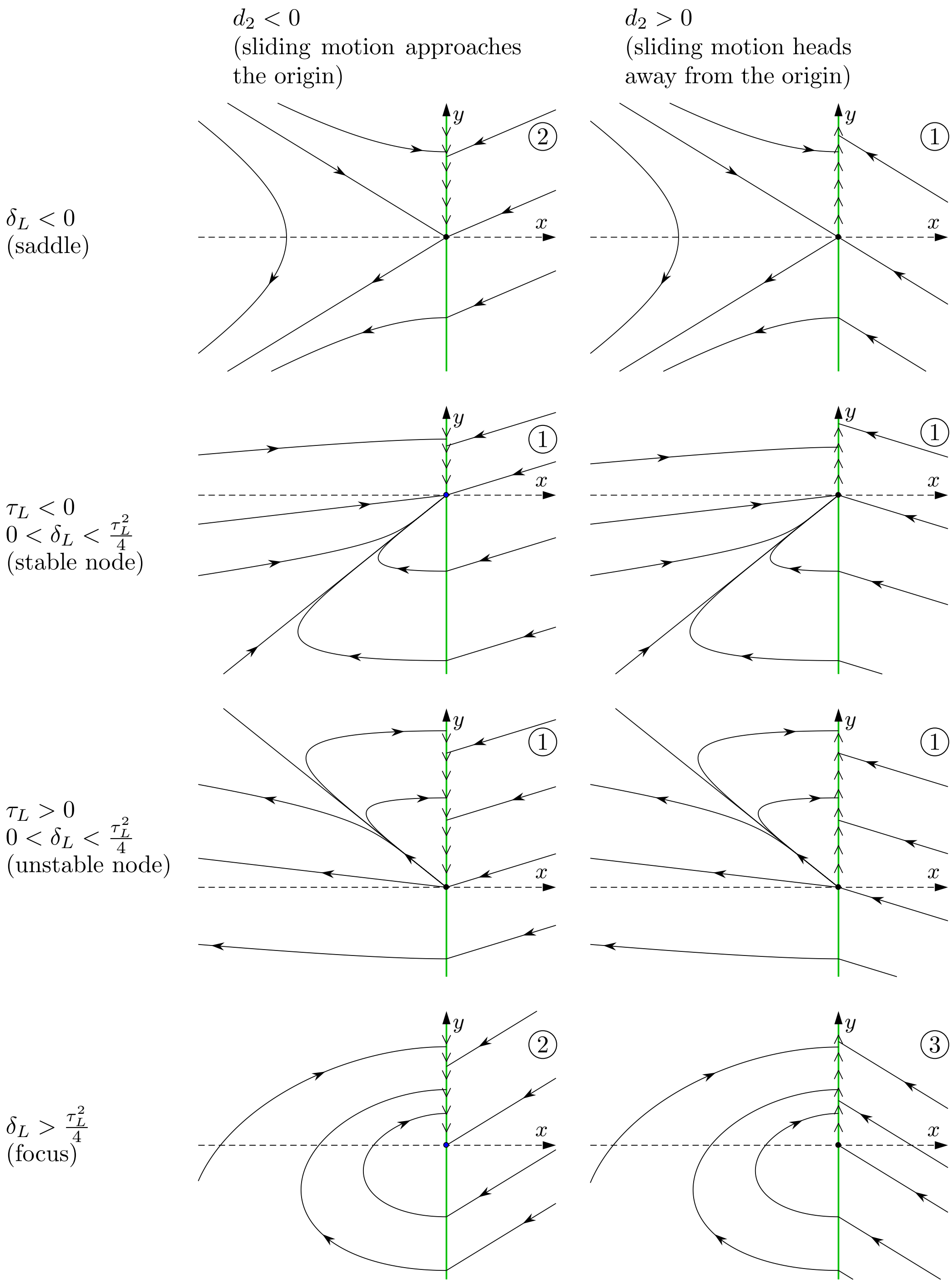

Figure 6: Filippov's eight scenarios for the dynamics local to a non-degenerate boundary equilibrium in two dimensions as exemplified by the normal form (8.15) with $d_1 = -1$. The circled numbers indicate the number of distinct unfoldings.

c) Now assume Q, r, and s are given by (8.9)–(8.11). Use the result from (b) to show that $QAQ^{-1} = C$.

d) Next show that $\frac{1}{s}\left(Qb - QAQ^{-1}r\right) = e_n$.

e) Finally show that the first component of $d = Qc$ is ± 1.

Exercise 8.5. Use the Cayley-Hamilton theorem to show that

$$s = \frac{(-1)^{n+1}}{|e_1^{\mathsf{T}} c|} \varrho^{\mathsf{T}} b, \tag{8.20}$$

where s is given by (8.11) and $\varrho^{\mathsf{T}} = e_1^{\mathsf{T}} \mathrm{adj}(A)$. Hint: you'll need to assume $\det(A) \neq 0$. The result can be extended to the case $\det(A) = 0$ by observing that ϱ is a continuous function on the space of $n \times n$ matrices, and that invertible matrices are dense in this space.

Exercise 8.6. Sketch typical phase portraits of the two-dimensional normal form (8.15) with $\delta_L > \frac{\tau_L^2}{4}$, $d_2 > 0$, and $\mu = \pm 1$. Make sure to explore all possibilities. Then, in about three sentences, summarise the three types of BEB that unfold the bottom-right boundary equilibrium of Fig. 6. Observe that the BEB (a *local* phenomenon) is partly governed by the *global* dynamics of a piecewise-linear system.

Exercise 8.7. Consider the predator-prey model of [28]:

$$\begin{aligned} \dot{x} &= x(1-x) - \frac{axy}{b+x}, \\ \dot{y} &= \frac{axy}{b+x} - D(y)y, \end{aligned} \tag{8.21}$$

where

$$D(y) = \begin{cases} d, & y < \alpha, \\ d+e, & y > \alpha. \end{cases} \tag{8.22}$$

In this model, $x(t)$ represents the prey population, $y(t)$ represents the predator population, and $a, b, d, e, \alpha > 0$ are parameters. The model assumes predators have a natural death rate of d, and are harvested at a rate e if they are sufficiently abundant ($y > \alpha$). To complete the following problems you may make reasonable additional assumptions on the values of the parameters.

a) Show that (8.21) has a BEB at $\alpha = \frac{b(a-d-bd)}{(a-d)^2}$.

b) Characterise this BEB: describe the associated invariant sets.

c) Show that two pseudo-equilibria collide and annihilate in a (classical) saddle-node bifurcation at $\alpha = \frac{(b+1)^2}{4a}$.

9 BEBs in continuous systems

- Now consider a system

$$\dot{\mathbf{x}} = \begin{cases} f_L(\mathbf{x};\mu), & H(\mathbf{x}) \le 0, \\ f_R(\mathbf{x};\mu), & H(\mathbf{x}) \ge 0, \end{cases} \tag{9.1}$$

  that is continuous on the switching manifold $H(\mathbf{x}) = 0$, call it Σ. That is, $f_L(\mathbf{x};\mu) = f_R(\mathbf{x};\mu)$ for all $\mathbf{x} \in \Sigma$.

- As in §7, assume $H(\mathbf{0}) = 0$ and $\zeta = H(\mathbf{0}) \neq \mathbf{0}$. Suppose f_L has an equilibrium at $\mathbf{x} = \mathbf{0}$ when $\mu = 0$. The continuity assumption implies

$$f_L(\mathbf{x};\mu) = A_L\mathbf{x} + b\mu + \mathcal{O}(2), \tag{9.2}$$
$$f_R(\mathbf{x};\mu) = A_R\mathbf{x} + b\mu + \mathcal{O}(2), \tag{9.3}$$

  for some $b \in \mathbb{R}^n$ and $n \times n$ matrices A_L and A_R that satisfy

$$A_R - A_L = c\zeta^{\mathsf{T}}, \tag{9.4}$$

  for some $c \in \mathbb{R}^n$.

- Locally, equilibria are given by

$$\mathbf{x}_L(\mu) = -A_L^{-1}b\mu + \mathcal{O}\left(\mu^2\right), \tag{9.5}$$
$$\mathbf{x}_R(\mu) = -A_R^{-1}b\mu + \mathcal{O}\left(\mu^2\right), \tag{9.6}$$

  assuming $\det(A_L) \neq 0$ and $\det(A_R) \neq 0$.

- To investigate the admissibility of $\mathbf{x}_L$ and $\mathbf{x}_R$, we write

$$H(\mathbf{x}_L(\mu)) = \alpha_L\mu + \mathcal{O}\left(\mu^2\right), \tag{9.7}$$
$$H(\mathbf{x}_R(\mu)) = \alpha_R\mu + \mathcal{O}\left(\mu^2\right). \tag{9.8}$$

  If $\alpha_L\alpha_R > 0$ [$\alpha_L\alpha_R < 0$], then $\mathbf{x}_L$ and $\mathbf{x}_R$ are admissible for different signs of μ [the same sign of μ] and the BEB is referred to as *persistence* [a *nonsmooth-fold*].

- Analogous to §7, we have

$$\alpha_L = -\frac{\varrho^{\mathsf{T}} b}{\det(A_L)}, \tag{9.9}$$

where we define

$$\varrho^{\mathsf{T}} = \zeta^{\mathsf{T}} \mathrm{adj}(A_L). \tag{9.10}$$

As shown in Exercise 9.2, the condition (9.4) implies

$$\zeta^{\mathsf{T}} \mathrm{adj}(A_R) = \zeta^{\mathsf{T}} \mathrm{adj}(A_L), \tag{9.11}$$

and thus

$$\alpha_R = -\frac{\varrho^{\mathsf{T}} b}{\det(A_R)}. \tag{9.12}$$

- As in the Filippov case, $\varrho^{\mathsf{T}} b \neq 0$ is the transversality condition for the BEB. Also, (9.9) and (9.12) give us the following Feigin-type result.

Theorem 9.1. *Suppose* $\det(A_L) \neq 0$, $\det(A_R) \neq 0$, *and* $\varrho^{\mathsf{T}} b \neq 0$. *Then*

$$\mathrm{sgn}(\alpha_L \alpha_R) = (-1)^{N_L + N_R}, \tag{9.13}$$

where N_L *is the number of real positive eigenvalues of* A_L, *and* N_R *is the number of real positive eigenvalues of* A_R.

Exercise 9.1. Consider the system

$$\begin{aligned} \dot{x} &= |x + y + \xi|, \\ \dot{y} &= x + 2y. \end{aligned} \tag{9.14}$$

Use Theorem 9.1 to classify the BEB at $\xi = 0$.

Exercise 9.2. Here you will derive (9.11).

a) Apply the matrix determinant lemma (7.12) to (9.4) to show that if $\det(A_L) \neq 0$ then

$$\det(A_L) \zeta^{\mathsf{T}} A_L^{-1} A_R = \det(A_R) \zeta^{\mathsf{T}}. \tag{9.15}$$

b) Use this to obtain (9.11) assuming also $\det(A_R) \neq 0$.

c) Use a density/continuity argument (see Exercise 8.5) to argue that (9.11) holds even if $\det(A_L) = 0$ or $\det(A_R) = 0$.

10 The normal form for BEBs in continuous systems

- Here we consider systems of the form

$$\dot{\mathbf{x}} = \begin{cases} A_L \mathbf{x} + b\mu + \mathcal{O}(2), & x \le 0, \\ A_R \mathbf{x} + b\mu + \mathcal{O}(2), & x \ge 0, \end{cases} \tag{10.1}$$

which have a BEB at $\mu = 0$. By continuity, A_L and A_R only differ in their first columns.

- Analogous to §8, for $Z \in \{L, R\}$, let

$$\Phi_Z = \begin{bmatrix} e_1^{\mathsf{T}} \\ e_1^{\mathsf{T}} A_Z \\ \vdots \\ e_1^{\mathsf{T}} A_Z^{n-1} \end{bmatrix}, \quad \Psi_Z = \begin{bmatrix} 1 & & & \\ a_{Z,1} & 1 & & \\ \vdots & \ddots & \ddots & \\ a_{Z,n-1} & \cdots & a_{Z,1} & 1 \end{bmatrix} \tag{10.2}$$

where $a_{Z,1}, \ldots, a_{Z,n}$ are the coefficients of the characteristic polynomial of A_Z (matching (8.5)). It turns out that $\det(\Phi_L) = 0$ if and only if $\det(\Phi_R) = 0$, see [29]. Consequently, it is sensible to say that (10.1) is *observable* if $\det(\Phi_L) \neq 0$.

- Let

$$Q = \Psi_L \Phi_L, \tag{10.3}$$

$$r = J^{\mathsf{T}} Q b, \tag{10.4}$$

$$s = e_n^{\mathsf{T}} Q b. \tag{10.5}$$

Assuming $s \neq 0$ (equivalently $\varrho^{\mathsf{T}} b \neq 0$, see Exercise 8.5) and $\det(\Phi_L) \neq 0$, the coordinate change

$$\mathbf{x} \mapsto Q\mathbf{x} + r\mu, \qquad \mu \mapsto s\mu, \tag{10.6}$$

is invertible and transforms (10.1) to

$$\dot{\mathbf{x}} = \begin{cases} C_L \mathbf{x} + e_n \mu, & x \le 0, \\ C_R \mathbf{x} + e_n \mu, & x \ge 0, \end{cases} \tag{10.7}$$

plus higher order terms, see Exercise 10.1. Equation (10.7) is the *BEB normal form* for continuous systems [6, 30], where C_L and C_R are the companion matrices

$$C_Z = \begin{bmatrix} -a_{Z,1} & 1 & & \\ -a_{Z,2} & & \ddots & \\ \vdots & & & 1 \\ -a_{Z,n} & & & \end{bmatrix}, \tag{10.8}$$

for $Z \in \{L, R\}$.

- The normal form (10.7) has $2n$ parameters (not counting μ). By scaling, it suffices to consider $\mu \in \{-1, 0, 1\}$.

- In two dimensions we can write (10.7) as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \mu \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \mu \end{bmatrix}, & x \ge 0, \end{cases} \tag{10.9}$$

  and there are ten topologically distinct generic BEBs[8] [31, 32].

- If $\mathbf{x}_L$ and $\mathbf{x}_R$ are foci of opposite stability, then at the BEB a local limit cycle is created in generic situations (see Exercise 10.2). A limit cycle is also created if $\mathbf{x}_L$ and $\mathbf{x}_R$ are a focus and a node of opposite stability (see Exercise 10.3). If $\mathbf{x}_L$ and $\mathbf{x}_R$ are nodes of opposite stability, then no limit cycle is created locally but a large amplitude limit cycle may be created due to global properties of the system (e.g. a 'canard super-explosion' [33]).
- As in the Filippov case, in three dimensions (10.7) can exhibit chaos [34].

Exercise 10.1. Show that $QA_RQ^{-1} = C_R$, where Q is defined by (10.3). HINT: write $A_R - A_L = ce_1^{\mathsf{T}}$.

Exercise 10.2. Consider (10.9) with $\tau_L > 0$, $\delta_L > \frac{\tau_L^2}{4}$, $\tau_R < 0$, and $\delta_R > \frac{\tau_R^2}{4}$. Write the eigenvalues of A_L and A_R as $\lambda_L \pm \mathrm{i}\omega_L$ and $\lambda_R \pm \mathrm{i}\omega_R$, where $\lambda_L > 0$, $\omega_L > 0$, $\lambda_R < 0$, and $\omega_R > 0$.

a) Fix $\mu = 0$ and derive a Poincaré map on $x = 0$.

b) Use your result from (a) to show that orbits approach the origin if $\Lambda = \frac{\lambda_L}{\omega_L} + \frac{\lambda_R}{\omega_R} < 0$, and that orbits head away from the origin if $\Lambda > 0$.

c) Argue that a stable [unstable] limit cycle is created in the BEB at $\mu = 0$ if $\Lambda < 0$ [$\Lambda > 0$].

Exercise 10.3. Consider (10.9) at parameter values for which $\mathbf{x}_L$ is an admissible unstable focus, and $\mathbf{x}_R$ is a virtual stable node. By sketching phase portraits, argue that the system must have a stable limit cycle.

Exercise 10.4. Write the vector field (10.9) as $\dot{x} = F(x, y)$, $\dot{y} = G(x, y)$. Let Γ be a limit cycle of (10.9) and let $\Omega \subset \mathbb{R}^2$ be the region that it encloses.

a) Show that $\oint_\Gamma -G\,dx + F\,dy = 0$.

b) Use Green's theorem[9] to obtain $\tau_L S_L + \tau_R S_R = 0$, where S_L [S_R] denotes the area of the part of Ω that lies in $x < 0$ [$x > 0$].

c) Conclude that (10.9) cannot have a limit cycle if $\tau_L \tau_R > 0$.

Exercise 10.5. Consider the McKean neuron model [35, 36]:

$$\dot{v} = F(v) - w + I,$$
$$\dot{w} = b(v - cw),$$

where

$$F(v) = \begin{cases} -v, & v \le \frac{a}{2}, \\ v - a, & \frac{a}{2} \le v \le \frac{a+1}{2}, \\ 1 - v, & v \ge \frac{a+1}{2}, \end{cases}$$

and $a, b, c, I > 0$ are parameters.

a) Show that a BEB occurs at $I = \frac{a(c+1)}{2c}$.

b) Characterise this BEB for $a = \frac{1}{4}$, $c = \frac{1}{2}$, and all $b > 0$.

Exercise 10.6. Consider the abstract system studied in [37]:

$$\dot{u} = -\alpha + \frac{2}{15}\beta + v + \frac{1}{5}u^2 + u^3,$$
$$\dot{v} = -\frac{5}{4}\alpha + \frac{1}{6}\beta - \frac{3}{8}u + \frac{1}{10}(\beta - 1)v + \left| \frac{1}{8}u - \frac{1}{10}v \right|.$$

Show that (α, β)-parameter space has a curve of BEBs that passes through $(\alpha, \beta) = (0, 0)$. How does the BEB change type at $(\alpha, \beta) = (0, 0)$?

[8] Namely: (i) saddle $\leftrightarrow$ saddle, (ii) saddle, repelling node $\leftrightarrow \varnothing$, (iii) saddle, repelling focus $\leftrightarrow \varnothing$, (iv) saddle, repelling focus, stable limit cycle $\leftrightarrow \varnothing$, (v) attracting node $\leftrightarrow$ attracting node, (vi) attracting node $\leftrightarrow$ attracting focus, (vii) attracting node $\leftrightarrow$ repelling focus, stable limit cycle, (viii) attracting node $\leftrightarrow$ repelling node, (ix) attracting focus $\leftrightarrow$ attracting focus, (x) attracting focus $\leftrightarrow$ repelling focus, stable limit cycle. All other cases can be obtained by reversing time.

[9] $\oint_\Gamma P\,dx + Q\,dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$.

11 Regular grazing bifurcations

- As the parameters of a piecewise-smooth ODE system are varied, a limit cycle can collide with a switching manifold. This is a called a *grazing bifurcation*, or a *sliding bifurcation* if sliding motion is involved.
- There are several types of grazing and sliding bifurcations [38]. These can be understood by constructing Poincaré maps. This is done quite generally in [6]. Here we construct Poincaré maps for two types of grazing and sliding bifurcations in three-dimensional systems for which coordinates are chosen as conveniently as possible.
- First we examine *regular grazing bifurcations*. These occur when a limit cycle collides with a switching manifold in a Filippov system and no sliding motion is involved. The assumption that no sliding motion is involved is non-generic but often arises from physical assumptions (such as for mechanical systems with compliant impacts [39, 40]). Mechanical systems with essentially instantaneous impacts are often better modelled by hybrid systems for which grazing bifurcations have similar algebraic properties [41].
- Consider a Filippov system of the form

$$\dot{\mathbf{x}} = \begin{cases} f_L(\mathbf{x};\mu), & x < 0, \\ f_R(\mathbf{x};\mu), & x > 0, \end{cases} \tag{11.1}$$

  where $\mathbf{x} = (x, y, z)$, and let Σ denote the switching manifold. Suppose, for both $Z = L$ and $Z = R$, that $\operatorname{sgn}\big(e_1^{\mathsf{T}} f_Z(\mathbf{x};\mu)\big) = \operatorname{sgn}(y)$. This ensures there is no sliding motion.
- Further suppose $e_2^{\mathsf{T}} f_Z(\mathbf{0};0) < 0$ (see already Fig. 7). Then, at least locally (i.e. for $(\mathbf{x};\mu)$ near $(\mathbf{0};0)$), the z-axis is a line of visible folds for f_L and a line of invisible folds for f_R.
- Suppose that when $\mu = 0$, (11.1) has a limit cycle that passes through $\mathbf{x} = \mathbf{0}$ but otherwise lies entirely within $x < 0$. That is, (11.1) has a regular grazing sliding bifurcation at $\mu = 0$.
- Let Π be the plane $y = 0$. We use Π, not Σ, to construct a Poincaré map because, locally, orbits have transversal intersections with Π.
- For (11.1) with a value of μ near 0, Fig. 7 depicts a typical orbit passing near $\mathbf{x} = \mathbf{0}$. The orbit intersects Π at $\mathbf{x}_2$ and then again at $\mathbf{x}_7$. If $\check{P}$ denotes the Poincaré map defined on Π in the usual way, then $\mathbf{x}_7 = \check{P}(\mathbf{x}_2)$. However, $\check{P}$ has a complicated form and it turns out to be much better to build an alternate map in a manner pioneered by Nordmark [42].

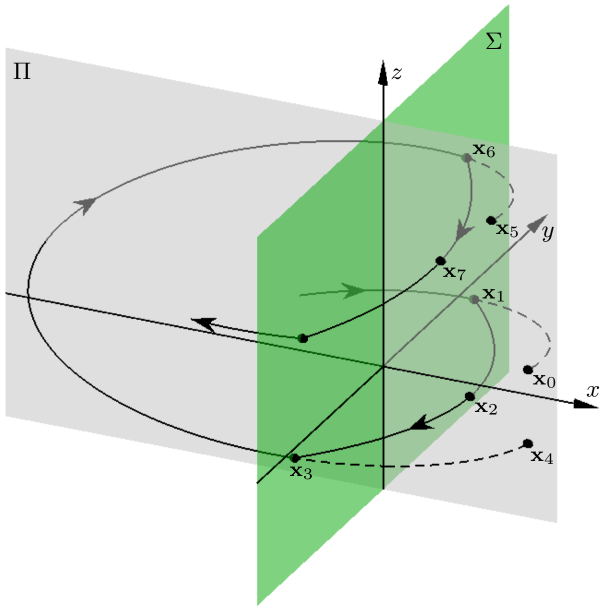

Figure 7: A sketch of an orbit of (11.1) near a regular grazing bifurcation.

- Near $\mathbf{x}_2$, the orbit intersects Σ first at $\mathbf{x}_1$ and then at $\mathbf{x}_3$. By following the flow of f_L forwards from $\mathbf{x}_1$ and backwards from $\mathbf{x}_3$, we obtain the virtual intersection points, $\mathbf{x}_0$ and $\mathbf{x}_4$, of the orbit with Π. We also follow the next intersection $\mathbf{x}_6 \in \Sigma$ forwards to the virtual point $\mathbf{x}_5 \in \Pi$.
- Let $P_{\rm global}$ be the Poincaré map on Π for f_L. That is, $P_{\rm global}$ takes $\mathbf{x}_4$ to $\mathbf{x}_5$. This map only involves f_L, so is smooth. Moreover, the limit cycle assumption implies $P_{\rm global}$ has the form

$$P_{\rm global}(x, z;\mu) = A \begin{bmatrix} x \\ z \end{bmatrix} + b\mu + \mathcal{O}(2), \tag{11.2}$$

  where $\mathcal{O}(2)$ represents terms that are quadratic or higher order in x, z, and μ.
- Let $P_{\rm disc}$ be the map that takes $\mathbf{x}_0$ to $\mathbf{x}_4$. This *discontinuity map* is a local map that provides the necessary correction to $P_{\rm global}$. If $x_0 \le 0$, we take $P_{\rm disc}$ to be the identity map. As you will

show in Exercise 11.1, $P_{\rm disc}$ has the form

$$P_{\rm disc}(x,z;\mu) = \begin{cases} \begin{bmatrix} x \\ z \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} x + \tilde{\mathcal{O}}(3) \\ z + \kappa\sqrt{x} + \tilde{\mathcal{O}}(2) \end{bmatrix}, & x \ge 0, \end{cases} \tag{11.3}$$

where $\tilde{\mathcal{O}}(k)$ represents terms that are order k or higher in $\sqrt{x}$, z, and μ.

- To analyse the grazing bifurcation we then study the composition $P = P_{\rm global} \circ P_{\rm disc}$. This map is continuous and has a square-root singularity. Its dynamics is described in, for instance, [6].

Exercise 11.1. Here you will derive the $x \ge 0$ part of (11.3).

a) Observe that the assumptions placed on f_L and f_R imply

$$f_L(\mathbf{x};\mu) = \begin{bmatrix} a_L x + b_L y + \mathcal{O}(2) \\ -c_L + \mathcal{O}(1) \\ d_L + \mathcal{O}(1) \end{bmatrix}, \tag{11.4}$$

$$f_R(\mathbf{x};\mu) = \begin{bmatrix} a_R x + b_R y + \mathcal{O}(2) \\ -c_R + \mathcal{O}(1) \\ d_R + \mathcal{O}(1) \end{bmatrix}, \tag{11.5}$$

where $b_L, c_L, b_R, c_R > 0$.

b) Let $\varphi_t^L(\mathbf{x};\mu)$ denote the flow of $\dot{\mathbf{x}} = f_L(\mathbf{x};\mu)$. Show that

$$\varphi_t^L(0,y,z;\mu) = \begin{bmatrix} b_L y t - \frac{b_L c_L}{2} t^2 + \mathcal{O}(3) \\ y - c_L t + \mathcal{O}(2) \\ z + d_L t + \mathcal{O}(2) \end{bmatrix}, \tag{11.6}$$

where $\mathcal{O}(k)$ represents terms that are order k or higher in y, z, μ, and t.

c) Use (11.6) to show that $\mathbf{x}_0$ is given in terms of $\mathbf{x}_1$ by

$$x_0 = \frac{b_L}{2c_L} y_1^2 + \mathcal{O}(3), \tag{11.7}$$

$$z_0 = z_1 + \frac{d_L}{c_L} y_1 + \mathcal{O}(2). \tag{11.8}$$

d) Similarly calculate $\mathbf{x}_3$ in terms of $\mathbf{x}_1$, and $\mathbf{x}_4$ in terms of $\mathbf{x}_3$.

e) Combine your results from (c)–(d) to produce (11.3) with

$$\kappa = \frac{2\sqrt{2c_L}}{\sqrt{b_L}} \left(\frac{d_R}{c_R} - \frac{d_L}{c_L} \right). \tag{11.9}$$

12 Grazing-sliding bifurcations

- A *grazing-sliding bifurcation* occurs when a limit cycle collides with a switching manifold Σ and the vector field on the 'other' side of Σ points towards Σ.

- Again we consider (11.1) and suppose $\mathrm{sgn}\big(e_1^{\mathsf{T}} f_L(\mathbf{x};\mu)\big) = \mathrm{sgn}(y)$, and $e_2^{\mathsf{T}} f_L(\mathbf{0};0) < 0$, so that the z-axis is a line of visible folds for f_L. Again suppose that when $\mu = 0$, (11.1) has a limit cycle that passes through $\mathbf{x} = \mathbf{0}$ but otherwise lies entirely within $x < 0$.

- Now, for a grazing-sliding bifurcation, suppose $e_1^{\mathsf{T}} f_R(\mathbf{0};0) < 0$.

- Fig. 8 shows a typical orbit. Let $P_{\rm global}$ be the Poincaré map on Π for f_L. This map has the form (11.2) and takes $\mathbf{x}_2$ to $\mathbf{x}_3$. Let $P_{\rm disc}$ be the map that takes $\mathbf{x}_0$ to $\mathbf{x}_2$ (and equal to the identity map if $x_0 \le 0$).

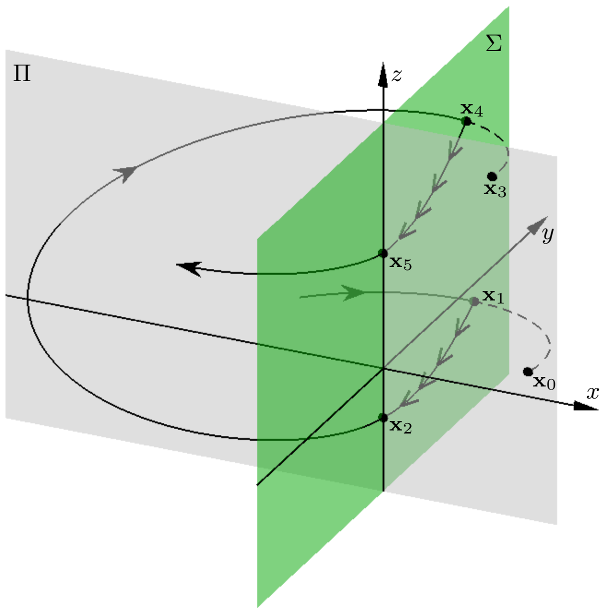

Figure 8: A sketch of an orbit of (11.1) near a grazing-sliding bifurcation.

- As you will show in Exercise 12.1, $P_{\rm disc}$ has the form

$$P_{\rm disc}(x,z;\mu) = \begin{cases} \begin{bmatrix} x \\ z \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} 0 \\ \kappa x + z + \mathcal{O}\left(\frac{3}{2}\right) \end{bmatrix}, & x \ge 0. \end{cases} \tag{12.1}$$

Interestingly the $\sqrt{x}$-terms in the maps from $\mathbf{x}_0$ to $\mathbf{x}_1$ and from $\mathbf{x}_1$ to $\mathbf{x}_2$ cancel out because, to leading order, the sliding vector field f_S (for motion from $\mathbf{x}_1$ to $\mathbf{x}_2$) is the same as f_L (for motion from $\mathbf{x}_1$ to $\mathbf{x}_0$) due to the tangency at $y = 0$. For this reason, $P = P_{\rm global} \circ P_{\rm disc}$ is continuous and piecewise-linear. Also notice that, for $x \ge 0$, the range of $P_{\rm disc}$ is one-dimensional and so here the Jacobian $DP(0,0;0)$ has a zero eigenvalue.

Exercise 12.1. Here you will derive the $x \ge 0$ part of (12.1).

a) Observe that the assumptions placed on f_L and f_R imply

$$f_L(\mathbf{x};\mu) = \begin{bmatrix} ax + by + \mathcal{O}(2) \\ -c + \mathcal{O}(1) \\ d + \mathcal{O}(1) \end{bmatrix}, \tag{12.2}$$

$$f_R(\mathbf{x};\mu) = \begin{bmatrix} -p + \mathcal{O}(1) \\ q + \mathcal{O}(1) \\ r + \mathcal{O}(1) \end{bmatrix}, \tag{12.3}$$

where $b, c, p > 0$.

b) Show that for $x = 0$ and $y > 0$,

$$f_S(\mathbf{x};\mu) = \begin{bmatrix} 0 \\ -c + \mathcal{O}(1) \\ d + \mathcal{O}(1) \end{bmatrix}. \tag{12.4}$$

c) Calculate $\mathbf{x}_0$ and $\mathbf{x}_2$ in terms of $\mathbf{x}_1$, and combine to obtain

$$P_{\rm disc}(x,z;\mu) = \begin{bmatrix} 0 \\ z + \tilde{\mathcal{O}}(2) \end{bmatrix}, \tag{12.5}$$

for $x \ge 0$, where $\tilde{\mathcal{O}}(2)$ denotes terms that are quadratic or higher order in $\sqrt{x}$, z, and μ. Argue that this implies the form (12.1).

d) Extra: derive a formula for κ in terms of the coefficients of f_L and f_R.

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