# A note on the charged isosceles three-body problem

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ABSTRACT: We study some aspects of a two-parameter family of ODE's which reduces, for $Q < 0$, to the isosceles three-body problem; for $Q = 0$, to the anisotropic Kepler problem; and, for $Q > 0$, to the charged isosceles three-body problem. In this case the collision manifold is the simplest possible, a sphere. It is well known that the collision-ejection orbits are a source of periodic orbits as the parameters change. We investigate, analytically and numerically, in the bifurcation diagram, a global connection between these orbits and the continuation of the classical circular Kepler orbit.

## 1. The Problem

Three particles move in a planar isosceles configuration as in figure 1. The two symmetric ones $q_1$ and $q_2$ have equal mass $M$ and equal charge $e$. The third particle, $q_3$, has mass $m$ and charge $f$ of opposite sign. Initial velocities are given so as to keep an isosceles configuration throughout the movement. $q_3$ has vertical initial velocity, $q_1$ and $q_2$ have initial velocities symmetric with respect to this vertical direction.

The equations of motion are given by Newton's law. Since the configuration remains isosceles, Jacobi coordinates are specially suitable for this problem (fig 1).

In these coordinates and after rescaling time (new time $\tau = t\sqrt{Gm - \frac{ef}{M}}$ ), the equations are ([A]):

$$\begin{cases} \ddot{x}_1 = -\dfrac{x_1}{r^3} + \dfrac{Q\ x_1}{|x_1|^3} \\ \ddot{x}_2 = -\mu \dfrac{x_2}{r^3} \end{cases}$$

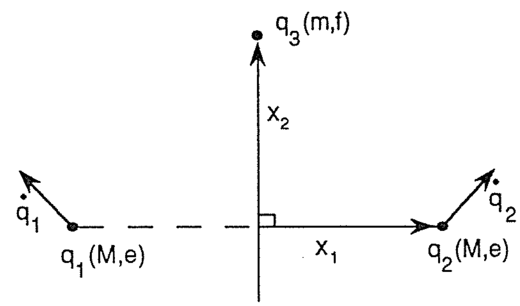


Fig. 1

with

$$\mu = \frac{2M + m}{m}, \qquad Q = \frac{e^2 - GM^2}{4(GMm - ef)},$$

where $r^2 = x_1^2 + x_2^2$, $\mu \geq 1$ is the mass ratio, and the parameter $Q$ measures the difference between gravitational and electrostatic forces. Since $e$ and $f$ have opposite signs, the denominator in $Q$ is always positive. For $Q < 0$ we have stronger gravitational forces and so particles $q_1$ and $q_2$ attract each other and double collisions are possible. $Q = -1$ is the classic isosceles 3-body problem. $Q = 0$ is the anisotropic Kepler problem (A.K.P) studied by Gutzwiller [1973], Devaney [1978], and Casasayas and Llibre [1984].

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For $(Q,\mu)=(0,1)$ we have the equations of the classical Kepler problem. For $Q>0$, the symmetric bodies repel each other and only triple collisions can occur. We will focus on this case.

We treat the two parameters $Q$ and $\mu$ as independent. Define the potential $V(x)$ and the mass matrix $M$ as

$$V(x)=\frac{-1}{(x_1^2+x_2^2)^{1/2}}+\frac{Q}{|x_1|}, \qquad M=\begin{pmatrix}1 & 0\\ 0 & \mu^{-1}\end{pmatrix},$$

where $x=(x_1,x_2)\in\mathbb{R}^2$. Taking $y=M\dot{x}$, the equations can be written in Hamiltonian form with Hamiltonian

$$H(x,y)=\frac{1}{2}y^tM^{-1}y+V(x).$$

## 2. Blow up, McGehee Coordinates and the Collision Manifold

We blow up the singularity at the origin with a change of coordinates due to McGehee [McG]. These are given by

$$\begin{aligned} r&=(x^tMx)^{1/2}, & v&=r^{1/2}s^ty,\\ s&=r^{-1}x, & u&=r^{1/2}(M^{-1}y-ys). \end{aligned}$$

It is easily seen that $s^tMs=1$, which defines an ellipse. Taking an angular coordinate $\theta=\tan^{-1}\frac{x_2}{x_1}$ on this ellipse, a new variable $\overline{u}$ such that $\overline{u}^2=u^tMu$, and rescaling time by a factor $r^{3/2}$, the equations of motion then become

$$\begin{cases} \dot{r}=rv\\ \dot{v}=u^2+\frac{1}{2}v^2+V(\theta)\\ \dot{\theta}=u\\ \dot{u}=-\frac{1}{2}uv-V'(\theta)\end{cases} \qquad \text{Energy integral:}\quad rh=\frac{1}{2}(u^2+v^2)+V(\theta) \qquad (2.1)$$

where $V(\theta)=\dfrac{-1}{\sqrt{\cos^2\theta+\mu\sin^2\theta}}+\dfrac{Q}{|\cos\theta|}$.

The so called collision manifold is the common intersection of all the energy level sets (2.1) with $r=0$. We will denote it by $\Lambda$. Notice that $r=0$ is an invariant set.

Using the energy integral, we reduce the system to determine a flow in $(u,\theta,v)\in\mathbb{R}^3$:

$$\begin{cases} \dot{v}=u^2+\frac{1}{2}v^2+V(\theta)\\ \dot{\theta}=u\\ \dot{u}=-\frac{1}{2}uv-V'(\theta).\end{cases} \qquad (2.2)$$

$\Lambda\subset\mathbb{R}^3$ is now a surface of revolution around the $\theta$-axis, invariant under the flow, with equation

$$\Lambda:\quad \frac{1}{2}(u^2+v^2)+V(\theta)=0.$$

We have three topologically different collision manifolds depending on the sign of $Q$. This bifurcation is presented in [A]. We focus in the case $0 < Q < 1$ where we obtain $\Lambda \cong S^2$, a 2-sphere.

For a fixed energy value $H = h < 0$, the energy level set is the interior of $\Lambda$. We denote this set by $E_h$. Thus, $\partial E_h = \Lambda$ and interior points correspond to real configurations $(x_1, x_2)$. All of $\Lambda$ corresponds to the triple collision point $x_1 = x_2 = 0$; orbits near $\Lambda$ are near collision.

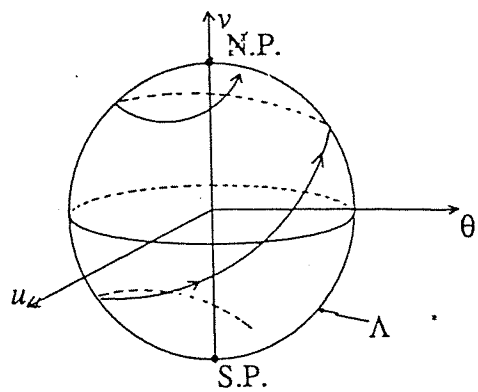


Fig. 2

## 3. The Flow

It is easily seen that there are no fixed points for the flow away from the collision manifold $\Lambda$. We summarize here its main features ([A]):

1. *The only two rest points are on the collision manifold $\Lambda$, they are the south and north poles.*
2. *On $\Lambda$, the flow spirals counterclockwise from the south pole to the north pole.*
3. *The vertical $v$-axis is an orbit and it is the only collision and/or ejection orbit.*
4. *The flow in $E_h$ circulates counterclockwise around the $v$-axis.*

The spiralling on $\Lambda$ is finite or infinite depending on the parameters. The effect of this on the periodic orbits bifurcating off the collision-ejection orbit is studied in [A]. This orbit is one of the "main sources" of periodic orbits. Here we show the "other source" which turns out to be the continuation of the classical circular Kepler orbit.

Let $\Sigma = \{u = 0, \theta > 0\} \cap E_h$. Except for the collision-ejection orbit, every orbit in $E_h$ crosses this 2-dimensional section transversally infinitely many times, i.e., it is a global Poincaré section. Let

$$P: \Sigma \to \Sigma$$

denote the corresponding first return (Poincaré) map. The existence of a fixed point for all values of the parameters can be shown by first looking at the flow on $\Lambda$. The image of the right-hand endpoint of $\Theta^+$ (the positive $\theta$-axis) under $P$ lies above the axis and the image of the left-hand endpoint ($\theta = 0$) lies under the axis. By continuity $P(\Theta^+)$

must intersect $\Theta^+$. By an index argument one can see that at least one of these points of intersection is a fixed point.

For $Q$ near 1 we now show that the intersection $P(\Theta^+) \cap \Theta^+$ is a single point.

As $Q \to 1$, the collision manifold reduces to a point and so does $E_h$. We study the limit flow as $Q \to 1$. Let $\epsilon = 1 - Q$. Expanding the potential $V$ in $\theta$ gives

$$V(\theta) = -\epsilon + \frac{\mu - \epsilon}{2}\theta^2 + \mathcal{O}(\theta^4, \epsilon\theta^4).$$

Thus, the collision manifold $\Lambda$ tends to the ellipsoid

$$u^2 + v^2 + \mu\theta^2 = 2\epsilon.$$

Expanding in powers of $\epsilon^{\frac{1}{2}}$:

$$\begin{aligned} v &= v_1\epsilon^{\frac{1}{2}} + v_2\epsilon + o(\epsilon) \\ \theta &= \theta_1\epsilon^{\frac{1}{2}} + \theta_2\epsilon + o(\epsilon) \\ u &= u_1\epsilon^{\frac{1}{2}} + u_2\epsilon + o(\epsilon). \end{aligned}$$

Substituting in eq. (2.2) and matching powers of $\epsilon$ gives

$$(*) \begin{cases} \dot{v}_1 = 0 \\ \dot{\theta}_1 = u_1 \\ \dot{u}_1 = -\mu\theta_1 \end{cases} \qquad (**) \begin{cases} \dot{v}_2 = u_1^2 - 1 + \dfrac{\mu\theta_1^2 + v_1^2}{2} \\ \dot{\theta}_2 = u_2 \\ \dot{u}_2 = -\dfrac{1}{2}u_1v_1 - \mu\theta_2. \end{cases}$$

Integrating the equations $(*)$ above and taking initial conditions $\left(0, \epsilon^{\frac{1}{2}}\theta_{10}, \epsilon^{\frac{1}{2}}v_{10}\right)$ on $\Sigma$ gives

$$\begin{aligned} v_1 &= v_{10}, \\ \theta_1 &= \theta_{10}\cos(\sqrt{\mu}t), \\ u_1 &= -\theta_{10}\sqrt{\mu}\sin(\sqrt{\mu}\,t). \end{aligned}$$

To leading order, the trajectories are ellipses; the map $P$ is then the identity. Substituting in $(**)$, integrating, and taking initial conditions zero gives

$$\begin{aligned} v_2(t) &= t(-1 + \frac{3}{4}\mu\theta_{10}^2 + \frac{1}{2}v_{10}^2) - \frac{1}{8}\theta_{10}^2\sqrt{\mu}\sin(2\sqrt{\mu}\,t), \\ \theta_2(t) &= \frac{1}{4}\theta_{10}v_{10}\left(\frac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}\,t) - t\cos(\sqrt{\mu}\,t)\right), \\ u_2(t) &= \frac{1}{4}\theta_{10}v_{10}\sqrt{\mu}\,t\sin(\sqrt{\mu}\,t). \end{aligned}$$

The time to return to $\Sigma$ is thus $2\pi/\sqrt{\mu} + o(\epsilon^{\frac{1}{2}})$, and the map $P$ is given by

$$\begin{pmatrix} \theta_{10} \\ v_{10} \end{pmatrix} \mapsto \begin{pmatrix} \theta_{10} \\ v_{10} \end{pmatrix} + \frac{2\pi}{\sqrt{\mu}} \epsilon^{\frac{1}{2}} \begin{pmatrix} -\frac{1}{4}\theta_{10}v_{10} \\ -1 + \frac{3}{4}\mu\theta_{10}^2 + \frac{1}{2}v_{10}^2 \end{pmatrix} + \mathcal{O}(\epsilon).$$

Thus, for sufficiently small $\epsilon$, $P$ has a unique fixed point $(\theta_{10}, v_{10}) = \left(\sqrt{\frac{4}{3\mu}} + \mathcal{O}(\epsilon^{\frac{1}{2}}), 0\right)$, which is elliptic with eigenvalues

$$\lambda_{1,2} = \exp\left(\pm i\pi\sqrt{2/\mu}\,\epsilon^{\frac{1}{2}} + \mathcal{O}(\epsilon)\right).$$

We find numerically that this point is the continuation of the circular Kepler orbit when $(Q, \mu) = (0, 1)$ (when the eqs. are identical to those of the classical two-body Kepler problem). We refer to it as the *Kepler orbit.*

As $Q$ decreases, periodic orbits bifurcate from the Kepler orbit. The generic bifurcations of an elliptic fixed point of an area-preserving diffeomorphism of the plane are well known: as $\lambda$ crosses an $n$th root of unity ($n \geq 5$), a pair of periodic orbits of period $n$ (one elliptic, one hyperbolic) emanates from the fixed point (Meyer [1970]).

$P$ is certainly the most natural Poincaré map for this problem. However, because of symmetries, $P$ has an area-preserving square root (the map $T$ below) for which the Kepler orbit is also a fixed point; hence the bifurcations of this fixed point of $P$ are not always the generic ones. The square root map is $T = S\hat{T}_{|\Sigma} : \Sigma \to \Sigma$ where $\hat{T}: \hat{\Sigma} \to \hat{\Sigma}$ is the first return map of $\hat{\Sigma} = \{u = 0, \theta \neq 0\} \cap E_h$ and $S: \hat{\Sigma} \to \hat{\Sigma}$ denotes reflection w.r.t. the $v$-axis.

If $\lambda, \lambda^{-1}$ are the eigenvalues of $T$ at the fixed point, we observe numerically that $\lambda$ evolves as follows: as the parameter $Q$ moves from one to zero, $\lambda$ moves around the unit circle monotonically from $\lambda = 1$ to $\lambda = -1$ and then becomes real. At $\lambda = -1$ a period doubling bifurcation for $T$ occurs and the Kepler orbit becomes unstable with $\lambda \in \mathbb{R}$. We also observe that when $\lambda = \exp(2\pi i m/n)$ and $n = 3$ or 4, the same bifurcation occurs as for $n > 4$; i.e., a pair of periodic orbits emanates from the Kepler orbit. We note that for $n = 3$ this is *not* the generic case treated in Meyer [1970].

Fig. 3 is computer generated showing the bifurcation diagram of orbits emanating from the Kepler orbit. The horizontal axis is the parameter $Q$. The vertical axis is the $\theta$ axis. $\mu = 1$ is fixed and as $Q$ decreases, orbits are "born" from the Kepler orbit and "die" at the collision-ejection orbit ($\theta = 0$). Each of these orbits can be labeled according to their rotation number (as points of the area preserving map $P$). The bifurcation occurring at the collision-ejection orbit was studied in [A]. At $Q = 1 - \frac{8}{9}\mu$, we have an $\infty$-furcation.

Fig. 4 shows different curves in the parameter space $(Q, \mu)$. Each dotted curve corresponds to parameter values for which we are at the "birth moment" of an orbit of a particular rotation number from the Kepler orbit. Fig. 3 shows the slice $\mu = 1$. The solid lines show bifurcations from the collision-ejection orbit: primarily the $\infty$-furcation, but for $\mu < \mu^* \sim 1.050055$, an orbit of rotation number $\frac{1}{2}$ under $P$ can be born first.

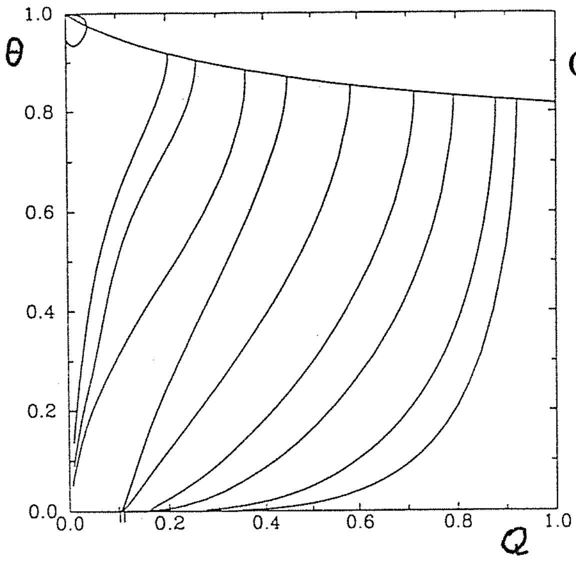


Fig. 3

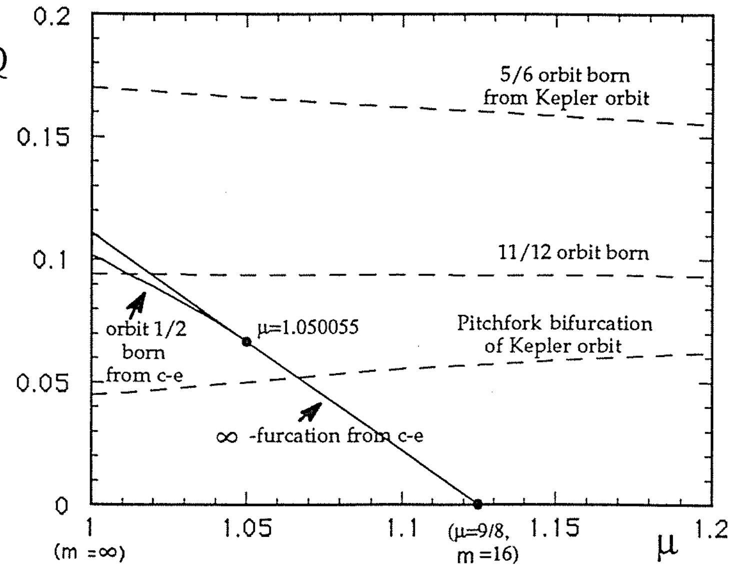


Fig. 4

Standard methods are not suitable for detecting and following these bifurcations, and we refer the reader to [A] for details.

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