Dynamics of the generalized Euler equations on Virasoro groups

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Abstract

We study the dynamics of the generalized Euler equations on Virasoro groups $\widehat{\mathcal{D}}(S^1)$ with different Sobolev H^k metric ($k \geq 2$) on the Virasoro algebra. We first prove that the solutions to generalized Euler equations will not blow up in finite time and then study the stability of the trivial solutions.

Key words Euler-Poincaré equation; Virasoro group; conjugate points.

1 Introduction and Main Results

Since Arnold's pioneering work [1], introducing the geometric approach to study the Euler fluid equation, many papers (such as [14], [16, 17, 18], [2] and the references therein) are devoted to this field.

Let $\mathcal{D}(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the unit circle S^1, then $\mathcal{D}(S^1)$ has a non-trivial one-dimensional central extension, the Bott-Virasoro group $\widehat{\mathcal{D}}(S^1)$ with the Virasoro algebra $\widehat{Vect}^s(S^1)$ ([17]), in which the commutator is given by

$$[\widehat{U}, \widehat{V}] \equiv \left((u_x v - u v_x)\frac{\partial}{\partial x},\ c(u,v) \right), \quad \text{with } c(u,v) \equiv \int_{S^1} u \partial_x^3 v \mathrm{d}x, \tag{1.1}$$

where $\widehat{U} = (u\frac{\partial}{\partial x}, a), \widehat{V} = (v\frac{\partial}{\partial x}, b) \in \widehat{Vect}^s(S^1)$ with $a, b \in \mathbb{R}$.

It is now well-known that the Euler equation on the Virasoro algebra for the L^2 metric (or equivalently, the geodesic equation on the Virasoro group of the right invariant metric which is L^2 metric at the identity) is the KdV equation ([19]), and the Euler equation for the H^1 metric is the Fuchssteiner-Fokas-Camassa-Holm equation ([4, 11, 14]). Recently, A.Constantin et al ([8, 9]) showed that on the Virasoro group, only the Euler equations for the L^2 metric and the H^1 metric are bi-Hamiltonian systems (see also [14]).

In order to study how the dynamics of the Euler equations depends on the different metric on the

Virasoro algebra, we are concerned in this paper with the generalized Euler equation

$$m_t + 2u_x m + u m_x = a\partial_x^3 u \qquad \text{on } S^1, \quad \text{with } m = A_k u, \tag{1.2}$$

where the operator $A_k = 1 - \partial_x^2 + \cdots + (-1)^k \partial_x^{2k}$, k is a positive integer and $a \in \mathbb{R}$.

For the local well-posedness of (1.2), we can apply the Kato theory ([13]) or the approach in ([10]) (see also ([15])). For Fuchssteiner-Fokas-Camassa-Holm equation equation (i.e., the Euler equation (1.2) with $k = 1$), a striking feature is that the solutions evolve into singularity in finite time if the initial momentum $m(0, x)$ assumes both positive and negative values. But for $k \geq 2$, our following result excludes this possibility.

Theorem 1 Suppose $k \geq 2$ in (1.2). If the initial value $m(0, x) \in L^2(S^1)$, then $m(t, x) \in L^2(S^1)$ for any finite time $t > 0$, and there exists a constant C_0 depending only on the initial condition such that

$$||m||_{L^2} \leq e^{C_0 t} ||m_0||_{L^2}. \tag{1.3}$$

Thereupon the global well-posedness of (1.2), and so it is of interest to study the geometry of the Virasoro group and consider the stability of some steady solutions. We can compute the sectional curvature and obtain

Theorem 2 The geodesic in $\widehat{\mathcal{D}}(S^1)$ with initial condition $\hat{\eta}(0) = (e, 0)$ and $\dot{\hat{\eta}}(0) = (v_0 \frac{\partial}{\partial x}, b)$, where v_0, b are constants, contains points conjugate to $\hat{\eta}(0)$ along $\hat{\eta}$.

2 Derivation of the equation (1.2)

We give a brief description how to get the Euler equation (1.2). Let $\widehat{U} = (u\frac{\partial}{\partial x}, a), \widehat{V} = (v\frac{\partial}{\partial x}, b), \widehat{W} = (w\frac{\partial}{\partial x}, c) \in \widehat{Vect}^s(S^1)$, and define the H^k inner product on $\widehat{Vect}^s(S^1)$ by

$$(\widehat{U}, \widehat{V})_{H^k} = \int_{S^1} (uv + u_x v_x + \cdots + \partial_x^k u \partial_x^k v)\mathrm{d}x + ab, \tag{2.1}$$

then we find $\mathrm{ad}^*_{\widehat{U}}$ by

$$\begin{aligned}
(\mathrm{ad}^*_{\widehat{U}} \widehat{V}, \widehat{W})_{H^k} &= (\widehat{V}, \mathrm{ad}_{\widehat{U}} \widehat{W})_{H^k} = (\widehat{V}, [\widehat{U}, \widehat{W}])_{H^k} \\
&= (v, u_x w - u w_x)_{H^k} + b \cdot c(u, w) \\
&= (g - b\partial_x^3 u, w)_{L^2} = (A_k^{-1}(g - b\partial_x^3 u), w)_{H^k},
\end{aligned} \tag{2.2}$$

where $g = 2u_x A_k v + u A_k v_x$. So

$$\mathrm{ad}^*_{\widehat{U}} \widehat{V} = \left(A_k^{-1}(2u_x A_k v + u A_k v_x - b\partial_x^3 u) \frac{\partial}{\partial_x}, 0 \right) \tag{2.3}$$

and the generalized Euler equation $\frac{\mathrm{d}}{\mathrm{dt}} \widehat{U} = -\mathrm{ad}^*_{\widehat{U}} \widehat{U}$ on the Virasoro group w.r.t. the right invariant metric gives us

$$\frac{\mathrm{d}A_k u}{\mathrm{d}t} = -(2u_x A_k u + u A_k u_x - a\partial_x^3 u), \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = 0, \tag{2.4}$$

which is (1.2) for $m = A_k u$.

3 Proof of the theorems

Proof of Theorem 1 We prove the Theorem 1 for sufficiently smooth function m and the general case $m_0 \in L^2$ follows by a standard density argument. Multiply (1.2) by m and integrate over S^1, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||m||_{L^2}^2 + 2\int u_x m^2 + \int umm_x = a\int m\partial_x^3 u, \tag{3.1}$$

Clearly, $\int m\partial_x^3 u = \int \partial_x^3 u A_k u = 0$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}||m||_{L^2}^2 = -3\int m^2 u_x, \tag{3.2}$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t}||m||_{L^2}^2 \le 3|u_x|_{L^\infty}||m||_{L^2}^2. \tag{3.3}$$

On the other hand, it is obvious that $H = \int_{S^1} um\mathrm{d}x$ is a conserved quantity for (1.2), i.e.

$$\sum_{l=0}^{k}||\partial_x^l u(t,x)||_{L^2}^2 = \sum_{l=0}^{k}||\partial_x^l u(0,x)||_{L^2}^2. \tag{3.4}$$

So from the Sobolev embedding theorem and $k \ge 2$ we have

$$|u_x|_{L^\infty} \le C||u_{xx}||_{L^2} \le C_0, \tag{3.5}$$

where C_0 is a constant depending only on the initial condition. The Gronwall inequality and (3.3) yield

$$||m||_{L^2} \le e^{C_0 t}||m_0||_{L^2}. \tag{3.6}$$

For the right invariant vector fields $\widehat{U}, \widehat{V}$, the covariant derivative $\nabla_{\widehat{U}}\widehat{V}$ can be obtained form the formular ([5])

$$\begin{aligned} 2\nabla_{\widehat{U}}\widehat{V} &= [\widehat{U},\widehat{V}] - \mathrm{ad}^*_{\widehat{U}}\widehat{V} - \mathrm{ad}^*_{\widehat{V}}\widehat{U} \\ &= [\widehat{U},\widehat{V}] - \Big((A_k^{-1}(2u_xA_kv + uA_kv_x + 2v_xA_ku + vA_ku_x - b\partial_x^3 u - a\partial_x^3 v)\tfrac{\partial}{\partial_x}, 0\Big) \\ &= \Big((u_xv - uv_x - A_k^{-1}(2u_xA_kv + uA_kv_x + 2v_xA_ku + vA_ku_x - b\partial_x^3 u - a\partial_x^3 v)\tfrac{\partial}{\partial_x}, c(u,v)\Big), \end{aligned} \tag{3.7}$$

so

$$\nabla_{\widehat{U}}\widehat{U} = -\left((A_k^{-1}(2u_xA_ku + uA_ku_x - a\partial_x^3 u)\frac{\partial}{\partial_x}, 0)\right).$$

On the other hand,

$$[[\widehat{U},\widehat{V}],\widehat{V}] = \left(((u_xv - uv_x)_x v - (u_xv - uv_x)v_x)\frac{\partial}{\partial x}, c(u_xv - uv_x, v)\right),$$

so by the formula ([5])

$$\begin{aligned} R(\widehat{U},\widehat{V}) &\equiv (R(\widehat{U},\widehat{V})\widehat{V},\widehat{U})_{H^k} = \tfrac{1}{4}||\mathrm{ad}^*_{\widehat{U}}\widehat{V} + \mathrm{ad}^*_{\widehat{V}}\widehat{U}||_{H^k}^2 - \Big(\mathrm{ad}^*_{\widehat{U}}\widehat{U},\ \mathrm{ad}^*_{\widehat{V}}\widehat{V}\Big)_{H^k} \\ &\quad -\tfrac{3}{4}||[\widehat{U},\widehat{V}]||_{H^k}^2 - \tfrac{1}{2}\Big([[\widehat{U},\widehat{V}],\widehat{V}],\widehat{U}\Big)_{H^k} - \tfrac{1}{2}\Big([[\widehat{V},\widehat{U}],\widehat{U}],\widehat{V}\Big)_{H^k}, \end{aligned} \tag{3.8}$$

we can get the sectional curvature formula although the calculation is lengthy and messy. However, if $\widehat{V}=(v_0\frac{\partial}{\partial x},b)$ is a constant vector field, then the direct computation gives the simple sectional curvature formula

$$\begin{aligned} R(\widehat{U},\widehat{V}) &= \frac{1}{4}b^2\int_{S^1}\partial_x^3 uA_k^{-1}\partial_x^3 u+v_0^2\int u_xA_k^{-1}u_x+bv_0\int u_{xx}A_k^{-1}u_{xx} \\ &=\frac{1}{4}\int_{S^1}\left(bA_k^{-1/2}\partial_x^3 u+2v_0A_k^{-1/2}u_x\right)^2\mathrm{d}x\geq 0, \end{aligned} \tag{3.9}$$

and the Riemannian curvature

$$\begin{aligned} R(\widehat{U},\widehat{V})\widehat{V} &= \nabla_{\widehat{U}}\nabla_{\widehat{V}}\widehat{V}-\nabla_{\widehat{V}}\nabla_{\widehat{U}}\widehat{V}-\nabla_{[\widehat{U},\widehat{V}]}\widehat{V} \\ &=\left((-\frac{1}{4}b^2A_k^{-2}\partial_x^6 u-v_0^2A_k^{-2}u_{xx}+bv_0A_k^{-2}\partial_x^4 u)\frac{\partial}{\partial x},0\right). \end{aligned} \tag{3.10}$$

Let $\widehat{\eta}(t)$ be the geodesic with the initial condition $\dot{\widehat{\eta}}(t)=\widehat{V}$, and $\widehat{W}(t)$ be an arbitrary vector along $\widehat{\eta}(t)$ and

$$(w(t,x)\frac{\partial}{\partial x},s(t))\equiv \mathrm{d}_{\widehat{\eta}(t)}R_{\widehat{\eta}^{-1}(t)}\widehat{W}(t),$$

where R_g denote the right multiplication by g on the Virasoro group. Then the Jacobi equation along $\widehat{\eta}(t)$

$$\nabla_{\dot{\widehat{\eta}}(t)}\nabla_{\dot{\widehat{\eta}}(t)}\widehat{W}(t)+R(\widehat{W}(t),\dot{\widehat{\eta}}(t))\dot{\widehat{\eta}}(t)=0 \tag{3.11}$$

reads $s''(t)=0$ and

$$\frac{\partial^2 w}{\partial t^2}-2v_0\frac{\partial^2 w}{\partial t\partial x}+v_0^2\frac{\partial^2 w}{\partial x^2}+2v_0^2A_k^{-1}\frac{\partial^2 w}{\partial x^2}-2v_0A_k^{-1}\frac{\partial^2 w}{\partial t\partial x}+bA_k^{-1}\frac{\partial^4 w}{\partial t\partial x^3}-bv_0A_k^{-1}\frac{\partial^4 w}{\partial x^4}=0 \tag{3.12}$$

that is

$$\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial x}\right)^2 w-2v_0A_k^{-1}\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial x}\right)w_x+bA_k^{-1}\left(\frac{\partial}{\partial t}-v_0\frac{\partial}{\partial x}\right)\partial_x^3 w=0. \tag{3.13}$$

For any integer $n\geq 1$, if we denote $k(n)=(1+n^2+n^4+\cdots+n^{2k})^{-1}$,

$$\mu=nv_0k(n)+\frac{1}{2}bk(n)n^3 \quad \text{and} \quad \lambda=nv_0+nv_0k(n)+\frac{1}{2}bk(n)n^3,$$

then a direct calculation tells us that

$$w(t,x)=\sin(\mu t)\sin(nx+\lambda t)$$

is a non-trivial solution to the Jacobi equation (3.12). And clearly, $\widehat{W}$ is always perpendicular to $\dot{\widehat{\eta}}(t)$, so it is a Jacobi field along $\widehat{\eta}(t)$. And if we take

$$t=\frac{2\pi j}{\mu} \quad \text{for } j=0,\pm1,\pm2,\cdots$$

we got the points conjugate to $\widehat{\eta}(0)$, which completes the proof of Theorem 2.

4 Comments

The solutions to the generalized Euler equation on the Virasoro group with different Sobolev H^k metric evolve non-integrably for $k \geq 2$. Due to the smoothing effect of the operator A_k^{-1}, the solution will not blow up in any finite time, contrast to the KdV equation and the Fuchssteiner-Fokas-Camassa-Holm equation. However, our numerical simulation indicates that the solution m does increase exponentially in the time t for some initial $m(0,x)$ assuming both positive and negative values.

Theorem 2 tells us that the constant solutions to (1.2) stay in the nonnegative sectional curvature region and contain some conjugate points. On the other hand, it is easy to see that the linearized equation of (1.2) at $\widehat{V} = (v_0 \frac{\partial}{\partial x}, b)$ is

$$Z_t + v_0 Z_x + 2v_0 A_k^{-1} Z_x = b A_k^{-1} \partial_x^3 Z, \tag{4.1}$$

and the quadratic

$$H(Z(t,x)) = v_0 \int_{S^1} Z(t, x+v_0 t) A_k^{-1} Z(t, x+v_0 t) + \frac{1}{2} b \int Z_x(t, x+v_0 t) A_k^{-1} Z_x(t, x+v_0 t) \tag{4.2}$$

is the Hamiltonian functional of (4.1), so it is conserved, from which we have the Eulerian linear stability at the trivial solutions to (1.2).

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