IOP Publishing European Journal of Physics

Eur. J. Phys. 37 (2016) 055003 (21pp) doi:10.1088/0143-0807/37/5/055003

Symmetry reduction for central force problems*

Robert I McLachlan[1,4], Klas Modin[2] and Olivier Verdier[3]

[1] Institute of Fundamental Sciences, Massey University, New Zealand
[2] Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, Sweden
[3] Department of Computing, Mathematics and Physics, Bergen University College, Norway

E-mail: r.mclachlan@massey.ac.nz, klas.modin@chalmers.se and olivier.verdier@math.uib.no

Received 10 January 2016, revised 2 May 2016
Accepted for publication 20 May 2016
Published 6 July 2016

Abstract

We give an elementary illustration of symmetry reduction for central force problems, drawing phase portraits of the reduced dynamics as the intersection of Casimir and energy level sets in three dimensions. These systems form a classic example of symplectic reduction which can usefully be compared to the more commonly seen case of the free rigid body.

Keywords: symplectic reduction, Hamiltonian systems, central force problems

(Some figures may appear in colour only in the online journal)

1. Introduction

Euler's equations for the motion of a free rigid body exemplify the power of the geometric viewpoint in classical mechanics and provide a first look into the merits and subtleties of symmetry. Indeed, Euler's rigid body equations are a perennially popular topic in mechanics, as they can be introduced and studied with little prior knowledge, yet present an array of advanced phenomena and techniques. They illustrate nearly all aspects of geometric mechanics and are repeatedly revisited each time new ideas are encountered. But, the almost exclusive focus on these equations does seem a little disproportionate. Why are the rigid body equations not one example amongst many, or, at least, the first in a sequence of increasing complexity and new phenomena?

[*] Dedicated to the memory of Jerry Marsden, 1942–2010.
[4] Author to whom any correspondence should be addressed.

Part of the reason is that the general theory of Hamiltonian systems with symmetry, exemplified by the rigid body equations, is thorny and still not fully understood. Complexities occur already in apparently simple situations, such as symmetries associated with the natural action of a matrix group on a vector space. Even the simplest general result, the *Marsden–Weinstein–Meyer symplectic reduction theorem*, is a topic for an advanced course in geometric mechanics, while attempts to cover more complicated, singular group actions, such as Ortega and Ratiu [21], are formidable and yet still incomplete.

In his lectures on physics [7, I, ch 22–1] Feynman rhetorically comments on why he has a chapter on algebra:

> Another reason for looking more carefully at algebra now, even though most of us studied algebra in high school, is that it was the first time we studied it; all the equations were unfamiliar, and it was hard work, just as physics is now. Every so often it is a great pleasure to look back to see what territory has been covered, and what the great map or plan of the whole thing is. Perhaps some day somebody in the Mathematics Department will present a lecture on mechanics in such a way as to show what it was we were trying to learn in the physics course!

In this paper we take a closer look at Hamiltonian symmetry reduction applied to equations that are intimately familiar to every physicist. First the rigid body equations, and then another example for which a concrete, elementary symplectic reduction can be carried out: the motion of a particle under a central force. This example duplicates and reinforces many of the features of the rigid body equations, and, like them, describes a physical system of fundamental importance. At the same time, the central force example provides a glimpse of a new structure of widespread importance in mathematical physics, namely the *dual pair*. A special study of the 'reduced central force equations' thereby provides a balancing perspective to the rigid body equations.

In our treatments of the free rigid body (section 2) and the reduction and reconstruction of central force problems (sections 3.1 and 3.2) we have aimed to be as elementary as possible; these sections are suitable for a first course in mechanics, the required background being Hamilton's equations and the chain rule. Sections 3.3 (dual pairs) and 3.4 (more bodies in more dimensions) are more advanced and are suitable for students of geometric mechanics.

2. The rigid body equations

The Euler equations for the motion of a triaxial free rigid body are

$$\begin{aligned}\dot{m}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right)m_2 m_3\\ \dot{m}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right)m_3 m_1\\ \dot{m}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right)m_1 m_2,\end{aligned} \tag{1}$$

where $I_1 \geqslant I_2 \geqslant I_3 > 0$ are the principal moments of inertia of the body, and m_i is the angular momentum about the ith principal axis of the body. The total angular momentum

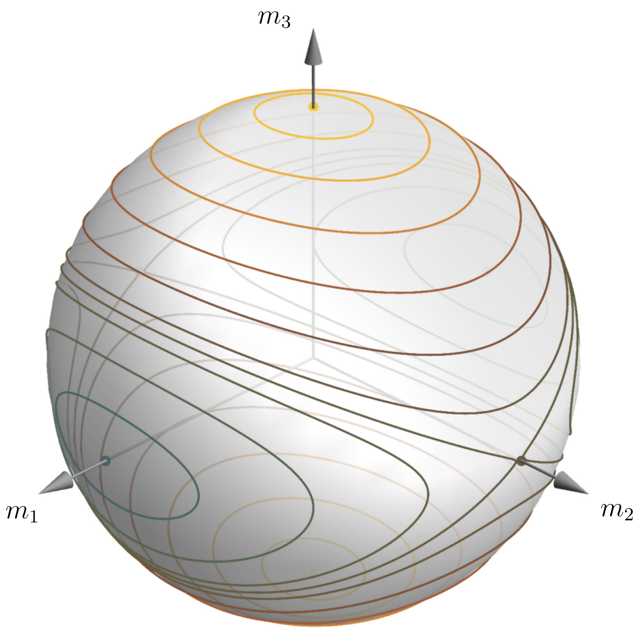

Figure 1. Phase portrait of the free rigid body. The sphere represents one level set of the angular momentum $C(m)$. The curves on the sphere represent several level sets of the Hamiltonian $H(m)$ restricted to the angular momentum sphere. Because both the angular momentum and the Hamiltonian are conserved, these level sets constitute the phase portrait. Notice that there are six relative equilibria: $(\pm m_1, 0, 0)$, $(0, \pm m_2, 0)$, and $(0, 0, \pm m_3)$. Each pair correspond to rotation, in positive or negative direction, about a principal axis. From the diagram one directly makes out that rotations about the m_1- and m_3-axes are stable, whereas rotations about the m_2-axis are unstable. The hammer throw experiment, illustrated in figure 2, is an easy way to demonstrate this phenomenon.

$$C(m) := m_1^2 + m_2^2 + m_3^2 \tag{2}$$

and the kinetic energy

$$H(m) := \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right) \tag{3}$$

are conserved quantities. Consequently, the phase portrait of (1) is given by the intersection of the spheres, determined by $C = \text{const.}$, and the ellipsoids, determined by $H = \text{const.}$, as shown in figure 1. This phase portrait shows that when the moments of inertia are distinct, i.e. when $I_1 > I_2 > I_3$, rotation about the m_1- and m_3-axes (the fixed points $(\pm m_1, 0, 0)$ and $(0, 0, \pm m_3)$, respectively) are stable, while rotation about the intermediate m_2-axis (the intermediate moment of inertia axis) is unstable. Indeed, rotations that start near $(0, m_2, 0)$ will follow a so called *heteroclinic orbit* to a neighbourhood of $(0, -m_2, 0)$. This can be illustrated in practice using the famous hammer throw experiment, shown in figure 2. In fact the experiment illustrates even more; in the course of traversing the heteroclinic orbit, the

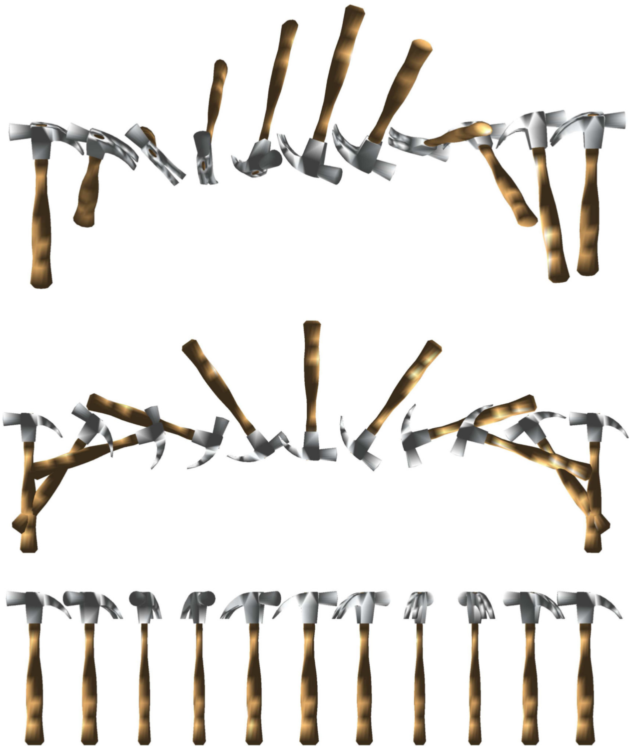

Figure 2. Simulation of the hammer throw experiment. The rotation of a hammer is shown when it is launched with an initial rotation about each of its three principal axes. The top rotation is unstable, resulting in the hammer undergoing a flip and landing in the opposite orientation, while the other two are stable.

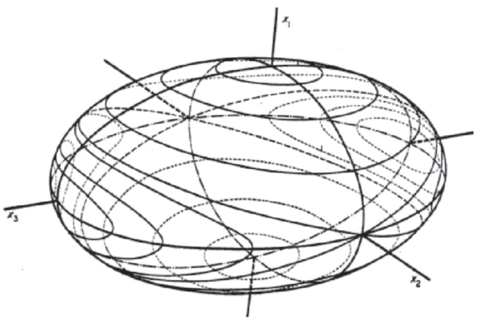

Figure 3. The phase portrait of the Euler equations of the free rigid body as drawn in Landau and Lifshitz's *Mechanics* [13].

attitude of the hammer undergoes a rotation by π—an example of a geometric (or Hannay–Berry) phase[5].

> Although most undergraduate physics text derive the Euler equations, few show the phase portrait as in figure 1. Usually, an analytic approach is used to study small oscillations about the stable rotation axes or to consider only symmetric rigid bodies, those with $I_1 = I_2$. One classic physics text that does include the phase portrait is Landau and Liftshitz, *Mechanics* [13], first published in Russian in 1940 and in English in 1960. Their version is shown in figure 3. Another (although hardly a standard undergraduate text) is Arnold's *Mathematical Methods of Classical Mechanics* [2]—see figure 121. (Arnold draws a constant-energy ellipsoid, instead of a constant-angular-momentum sphere.) A mathematics text that includes the phase portrait is Bender and Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, figure 4.31 [4].

A more detailed treatment would view equation (1) as the result of a *symmetry reduction*. Indeed, symmetry has a claim to be the single most important unifying and organising principle in physics. The *attitude* of a free rigid body is specified by a rotation matrix Q, which gives the rotation of the body from an initial reference position. Thus, the configuration space of the body is the set of proper 3×3 orthogonal matrices

$$\mathrm{SO}(3) \coloneqq \{Q \in \mathbb{R}^{3\times 3} := Q^T Q = I,\ \ \det Q = 1\}.$$

Its phase space is the space of possible attitudes together with the angular momenta of the body, whose points can be written as (Q, m), where $Q \in \mathrm{SO}(3)$ is the attitude of the body and $m \in \mathbb{R}^3$ is its angular momentum in the 'body frame', that is, in coordinates that are fixed in the body and rotate along with it[6]. If $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector in the body frame and $m = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$ is the corresponding angular momentum, then the full, unreduced equations of motion are

$$\dot{Q} = Q\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \eqqcolon Q\tilde{\omega}, \tag{4}$$

$$\dot{m} = m \times \omega. \tag{5}$$

Let us now discuss symmetry reduction of these equations. First, the space $\mathrm{SO}(3)$ of rotation matrices form a *group*: if A and B are rotation matrices then their product AB is again a rotation matrix and every rotation matrix has an inverse[7]. Second, to each element of the group is associated a transformation of phase space; we say that the group *acts* on phase space. Here the group $\mathrm{SO}(3)$ acts on phase space by the transformations

$$A \cdot (Q, m) \ := \ (AQ, m). \tag{6}$$

[5] Many people know this trick, but we suspect that not many could explain why the rotation angle is π. For a complete explanation of why a rotation by π is observed for 'hammer-like' objects, see Cushman and Bates [5, III.8].

[6] A more classical formulation, originally by Euler, is to use three Euler angles for the attitude together with three conjugate momenta—six variables altogether. However, the Euler angles do not cover all possible attitudes of the body and are singular near some attitudes.

[7] More precisely, $\mathrm{SO}(3)$ is a *Lie group*, which roughly means that it can be thought of as a smooth 'surface' in the space of all 3×3 matrices, and that multiplication and inversion are infinitely differentiable operations. In this paper it suffices to think of a Lie group as a subset of invertible matrices closed under matrix multiplication.

In geometry and physics, *symmetry* means invariance of the phase portrait under a group of transformations of phase space. The full rigid body system ((4) and (5)) is invariant under the group action (6): if $(Q(t), m(t))$ is a solution then $A \cdot (Q(t), m(t))$ is another solution[8]. More fundamentally, the symmetry of the system enters through the Hamiltonian; if $H(Q, m)$ is the Hamiltonian (kinetic energy in the case of the rigid body), symmetry means that $H(A \cdot (Q, m)) = H(Q, m)$ for any $A \in \mathrm{SO}(3)$. A consequence of this is three conserved quantities, namely the spatial angular momentum vector given by Qm. Indeed, if $Q = Q(t)$, $m = m(t)$ is a solution, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Qm &= \dot{Q}m + Q\dot{m} \\ &= Q\tilde{\omega}m + Q(m \times \omega) \\ &= Q\tilde{\omega}m - Q(\omega \times m) \\ &= Q\tilde{\omega}m - Q\tilde{\omega}m = 0. \end{aligned} \tag{7}$$

That continuous symmetries of the Hamiltonian (or Lagrangian) implies conservation laws is a fundamental principle in physics, first proved by the German mathematician Emmy Noether in 1915.

Due to the symmetry, the equations of motion ((4) and (5)) for the full rigid body can be *reduced*. Here is the basic idea: since the action of $\mathrm{SO}(3)$ on phase space maps solutions to solutions, we can 'quotient out' equivalent solutions, so that two points (Q, m) and $(\tilde{Q}, \tilde{m})$ in the original phase space are considered the same if there exists a rotation matrix A such that $A \cdot (Q, m) = (\tilde{Q}, \tilde{m})$. We say that (Q, m) and $(\tilde{Q}, \tilde{m})$ belong to the same *group orbit*. The *quotient space*, denoted $(\mathrm{SO}(3) \times \mathbb{R}^3)/\mathrm{SO}(3)$, is the set where each element is a group orbit. Symmetry reduction now consists in formulating the dynamics on the quotient space. But the quotient space is defined abstractly, so we need to equip it with coordinates in order to 'work with it'. This is not always easy, but for the rigid body equations it is trivial. Indeed, the orbit containing (Q, m) consists of the points $A \cdot (Q, m) = (AQ, m)$ for every $A \in \mathrm{SO}(3)$. In particular, taking $A = Q^{-1}$ we see that two points (Q, m) and $(\tilde{Q}, \tilde{m})$ belong to the same orbit if and only if $m = \tilde{m}$. In other words, we may use $m = (m_1, m_2, m_3)$ as coordinates for the quotient space. Since the full rigid body equations (4) and (5) have $\mathrm{SO}(3)$ as a symmetry group, we expect the dynamics to reduce to a system in m that is independent of Q. Voila! Equation (5) involves only m, so symmetry reduction in this case simply means to keep equation (5) and disregard equation (4). In the central force example, treated in section 3, it is more complicated to find coordinates for the quotient space.

We now come to the *geometry* of symmetry reduction, or more precisely, *symplectic geometry*. A Hamiltonian system in classical mechanics, in particular the full rigid body equations (4) and (5), conserve a *canonical symplectic structure* (see [2, ch 8]). In the simplest case of a two-dimensional phase space, it means that the phase flow preserves area. One may ask how the symplectic structure is carried along in symmetry reduction. In other words, does the reduced system preserve a symplectic structure? This question is at the very heart of geometric mechanics. Canonical Hamiltonian systems are always even-dimensional and hence the three-dimensional rigid body equations cannot possibly be canonical—we need a generalised structure. The relevant generalisation consists in *Poisson structures*. Indeed, in essence the fundamental result is that if the action of the symmetry group on the phase space preserves the symplectic structure (the action is

[8] We encourage the reader to verify this.

Hamiltonian), then the reduced system is a *noncanonical Hamiltonian system* with respect to a Poisson structure. This is manifested in the celebrated Marsden–Weinstein–Meyer symplectic reduction theorem [16, 18]. We now describe what it means in the concrete case of the rigid body.

Let $\mathcal{M}$ denote the reduced phase space of some system (for the rigid body $\mathcal{M} = \mathbb{R}^3$ as we have seen). A Poisson bracket $\{,\}$ on $\mathcal{M}$ is an operation that satisfies the axioms

$$\begin{aligned} \{,\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \\ \{F, G\} = - \{G, F\} \\ \{F, aG + bH\} = a\{F, G\} + b\{F, H\} \\ \{F, GH\} = \{F, G\}H + \{F, H\}G \end{aligned}$$

for all $F, G, H \in C^\infty(\mathcal{M})$ and for all $a, b \in \mathbb{R}$. Furthermore, a Poisson system with Hamiltonian H is an equation of the form

$$\dot{m} = K(m)\nabla H(m), \tag{8}$$

where the *Poisson tensor* K is defined by

$$\{F, G\} = (\nabla F)^T K(m)(\nabla G).$$

Every canonical system is a Poisson system, but not vice versa. In the case of the reduced rigid body equations (5)

$$K(m) = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \tag{9}$$

and the Hamiltonian is given by (3). We encourage the reader to check this.

If F is a function on $\mathcal{M}$, then along solutions to (8) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F = (\nabla F)^T \dot{m} = (\nabla F)^T K(m)\nabla H = \{F, H\}.$$

Thus, F is a first integral if and only if $\{F, H\} = 0$. Notice, in particular, that the Hamiltonian H is a first integral, since $\{H, H\} = 0$ due to skew-symmetry of the Poisson bracket.

For noncanonical Poisson structures there are special functions C fulfilling

$$\{C, H\} = 0 \quad \text{for any function } H.$$

They are called *Casimir functions* and are first integrals for any choice of Hamiltonian: they depend only on the Poisson structure. For the rigid body Poisson structure (9) there is one Casimir function given by the total angular momentum (2). The common interpretation in geometric mechanics is that the Casimir functions are associated with the geometry of phase space (they only depend on the Poisson structure), whereas the Hamiltonian (total energy) is associated with the particular dynamics of the system. Returning to figure 1, the sphere on which the phase portrait is drawn thereby reflects the geometry of phase space, whereas the individual curves are determined by the specific choice of Hamiltonian. Also notice that the entire phase space $\mathbb{R}^3$ is *foliated* in angular momentum spheres; it turns out that all Poisson structures are foliated in a similar fashion into so called *symplectic leaves* [27].

Poisson systems in which $\mathcal{M}$ is a vector space and $K(m)$ is a linear function of m play a special role. This case is called *Lie–Poisson*, because the Poisson structure originates from a *Lie algebra* [20]. Lie algebras are everywhere in modern physics. But what are they?

Geometrically, they are infinitesimal variations of Lie groups: the Lie algebra of SO(3), denoted $\mathfrak{so}(3)$, consists of skew-symmetric 3×3 matrices[9,10]. Algebraically, they are vector spaces equipped with *Lie brackets*: for $\mathfrak{so}(3)$ the Lie bracket is given by the matrix commutator $[a, b] = ab - ba$ (we encourage the reader to verify that if $a, b \in \mathfrak{so}(3)$ then $[a, b] \in \mathfrak{so}(3)$). Often, $\mathfrak{so}(3)$ is identified with $\mathbb{R}^3$ using the mapping $\tilde{\omega} \leftrightarrow \omega$ in (4). This makes $\mathbb{R}^3$ into a Lie algebra with bracket $[u, v] = u \times v$ (we encourage the reader to verify this).

In general, the equations of motion for an n-dimensional Lie–Poisson system with Hamiltonian H can be written

$$\dot{m}_i = \sum_{j,k=1}^{n} c_{ij}^{k} m_k \frac{\partial H}{\partial m_j}, \quad i = 1, \ldots, n.$$

Here, the real numbers c_{ij}^k are the *structure constants* of the Lie algebra, defined by

$$[u, v]_k = \sum_{i,j=1}^{n} c_{ij}^{k} u_i v_j$$

for all Lie algebra elements u and v. Another way of expressing this is to say that there are square matrices $M_1, \ldots, M_n$ whose Lie (commutator) brackets obey

$$[M_i, M_j] := M_i M_j - M_j M_i = \sum_{k=1}^{n} c_{ij}^{k} M_k. \tag{10}$$

Such matrices form a basis for the Lie algebra.

The reduced rigid body equations (5) are the motivating example of the large class of *Euler–Arnold equations* that describe geodesics on groups with respect to an invariant Riemannian metric. Such equations were advocated in 1966, when Arnold [1] discovered that the Euler rigid body and the Euler fluid equations share this basic structure, with the groups being SO(3) in the first case and the volume-preserving diffeomorphisms in the second case. Since then, the field has grown enormously, and now encompasses a multitude of equations in physics such as the Korteweg–de Vries, Camassa–Holm, and Landau–Lifshitz equations [12].

In summary, the rigid body equations provide an excellent example of (i) systems with nonlinear phase spaces; (ii) visualisation of phase portraits; (iii) noncanonical Hamiltonian systems; (iv) Poisson systems; (v) Lie–Poisson systems; (vi) Marsden–Weinstein–Meyer symplectic reduction; (vii) reconstruction and geometric phases; (viii) Casimirs; (ix) Euler–Arnold systems; (x) dynamical phenomena discovered using geometric methods, without solving any differential equations. Moreover, they apply to readily accessible everyday experiences, notably the hammer throw experiment and the rotation of the Earth[11].

9 It is not a coincidence that the matrix $\tilde{\omega}$ in equation (4) is skew-symmetric: it needs to be an element of the Lie algebra $\mathfrak{so}(3)$ in order for Q to belong to SO(3).

10 All groups and algebras in this paper are real; thus $\mathfrak{so}(3) = \mathfrak{so}(3, \mathbb{R})$, $\mathfrak{sp}(2) = \mathfrak{sp}(2, \mathbb{R})$, etc.

11 A topic of current research interest; see, e.g., [22].

3. Central force problems

3.1. The reduced central force equations

Consider a particle moving in three-dimensions subject to a conservative central force. It has position $q \in \mathbb{R}^3$, momentum $p \in \mathbb{R}^3$, and total (kinetic plus potential) energy

$$\widetilde{H}(q, p) = \frac{1}{2}\|p\|^2 + V(\|q\|^2). \tag{11}$$

The equations of motion are Hamilton's equations

$$\begin{aligned} \dot{q} &= \frac{\partial \widetilde{H}}{\partial p} = p, \\ \dot{p} &= -\frac{\partial \widetilde{H}}{\partial q} = -2V'(\|q\|^2)q. \end{aligned} \tag{12}$$

The phase space is $\mathcal{M} = \mathbb{R}^6$ and the Hamiltonian is orthogonally invariant. That is, the symmetry group consists of the orthogonal matrices

$$\mathrm{O}(3) \coloneqq \{A \in \mathbb{R}^{3\times 3} := A^T A = I\},$$

which acts on $\mathcal{M}$ by

$$A \cdot (q, p) = (Aq, Ap). \tag{13}$$

The symmetry applies the same rotation A to both the position q and the momentum p. The Hamiltonian (11) is invariant under the symmetry, i.e.

$$\widetilde{H}(Aq, Ap) = \widetilde{H}(q, p)$$

for all orthogonal matrices A. The equations of motion, equation (12), are orthogonally invariant as well.

Let us now 'quotient out' the symmetry group to obtain reduced equations and simple phase portraits. As for the rigid body, this will be done in three steps:

1. by passing to the space of orbits of G, yielding a three-dimensional Poisson system;
2. by finding the Casimir of this system and hence passing to the corresponding two-dimensional level set (called a symplectic leaf);
3. by obtaining the phase portrait as the level sets of the Casimir and the energy.

Each step is essentially geometric; the only calculations that are required involve the chain rule of multivariable calculus. We will show examples of diverse dynamical behaviour in central force problems that can be 'read off' without solving any differential equations.

In fact, as this procedure can be followed for *any* $\mathrm{O}(3)$-invariant Hamiltonian, we will carry it out for the general case (not just for Hamiltonians of the form (11)).

Step 1. Passing to the space of group orbits, yielding a three-dimensional Poisson system. If the Hamiltonian is $\mathrm{O}(3)$-invariant, that means it is constant on each orbit of the group. The group action, a common rotation or reflection of q and p, preserves $\|q\|^2$ and $\|p\|^2$, and it also preserves $q \cdot p$, essentially the angle between q and p. It is easy to see that given any two pairs (q, p) with the same values of $\|q\|^2$, $q \cdot p$, and $\|p\|^2$, there must be a symmetry operation—a common orthogonal transformation of q and p—that sends one pair into the other. That is, we can use the values of

$$\begin{aligned} w_1 &:= \|q\|^2, \\ w_2 &:= q \cdot p, \\ w_3 &:= \|p\|^2, \end{aligned} \tag{14}$$

as coordinates for the quotient space $\mathbb{R}^6/\mathrm{O}(3)$ of group orbits. We have thus found that any $\mathrm{O}(3)$-invariant Hamiltonian $\widetilde{H} = \widetilde{H}(q, p)$ can be written in the form

$$\widetilde{H}(q, p) = H(\|q\|^2, q \cdot p, \|p\|^2) \tag{15}$$

for some function $H = H(w_1, w_2, w_3)$.

The appearance of w_1, w_2, and w_3 is no surprise, as the orthogonal group $\mathrm{O}(3)$ can be *defined* to be the linear maps that preserve all dot products between vectors. However, although we reached (15) easily, there is much more going on here, which we will amplify in this aside.

The background to the group action (13) is the Euclidean metric on the space $\mathbb{R}^3$ of positions. The action $A \cdot q := Aq$ of $A \in O(3)$ on the position q is the isometry group of the metric. More generally, if the configuration space of a mechanical system is a Riemannian manifold Q, then its phase space T^*Q (its space of positions and momenta) carries a canonical Riemannian metric called the Sasaki metric [25, 26]. In coordinates (q_i, p_i), where $g_{ij}\mathrm{d}q_i \otimes \mathrm{d}q_j$ is the metric on Q, the Sasaki metric is given by $g_{ij}\mathrm{d}q_i \otimes \mathrm{d}q_j + (g^{-1})_{ij}\mathrm{d}p_i \otimes \mathrm{d}p_j$. The action of the isometry group of Q then lifts to a isometry (equation (13) in our case) of T^*Q.

The quadratics $\|q\|^2$, $q \cdot p$, and $\|p\|^2$ are *invariants* of the symmetry group. Any function of these invariants, like (15), is necessarily invariant. But much more is true. It is known for this symmetry group that *any* invariant polynomial in q, p must be a polynomial in the w_1, w_2, and w_3. This is an example of what is called a *first fundamental theorem* for a symmetry group; such a theorem is known in only a few cases. From there it is a small step to conjecture that if $\widetilde{H}(q, p)$ is a smooth invariant function, then $H(w_1, w_2, w_3)$ will be a smooth function as well. This turns out to be true. (*A priori*, one might have worried the w_i being quadratic functions of q and p might mean $H(w_1, w_2, w_3)$ could end up involving square roots. That cannot happen.)

This is an example of a very general and celebrated theorem of Schwartz [23] that states that for a compact group acting on a vector space, *any* smooth invariant function on the vector space is a smooth function of the invariants. In fact, it must be a smooth function of the polynomials that generate the invariant polynomials. In our case these generating polynomials are $\|q\|^2$, $q \cdot p$, and $\|p\|^2$.

While Hilbert's invariant theorem guarantees that for many (but not all) matrix groups the set of invariant polynomials is finitely generated, it may not be a simple manner to actually construct a generating set.

Furthermore, we showed above why the invariants in this example serve to uniquely label the orbits of the symmetry group. In other examples, even when a generating set for the invariants can be found, those invariants may not serve to uniquely label the the orbits. For example, the one-dimensional scaling group acting on the plane has no continuous invariants, because the orbits (open rays) have a common limit point, the origin. So the situation considered here is actually quite special.

For the Hamiltonian $\widetilde{H}(q, p) = H(\|q\|^2, q \cdot p, \|p\|^2) = H(w_1, w_2, w_3)$, Hamilton's equations are

$$
\begin{aligned}
\dot{q} &= \frac{\partial \widetilde{H}}{\partial p} &&= q\frac{\partial H}{\partial w_2} + 2p\frac{\partial H}{\partial w_3},\\
\dot{p} &= -\frac{\partial \widetilde{H}}{\partial q} &&= -2q\frac{\partial H}{\partial w_1} - p\frac{\partial H}{\partial w_2}.
\end{aligned} \tag{16}
$$

As $\|q\|^2 \geqslant 0$, $\|p\|^2 \geqslant 0$, and $\|q \times p\|^2 = \|q\|^2\|p\|^2 - (q \cdot p)^2 \geqslant 0$, not all values of (w_1, w_2, w_3) are realisable. Indeed, the space of group orbits is contained in the cone

$$\{(w_1, w_2, w_3) \in \mathbb{R}^3 := w_1 \geqslant 0,\ w_3 \geqslant 0,\ w_1 w_3 - w_2^2 \geqslant 0\}$$

and it is easy to check that it does consist of the entire cone. This situation, in which we have a complete description of the space of group orbits, with each orbit represented as a point in a vector space, is quite rare.

As we know that the flow of any G-invariant system, Hamiltonian or not, maps orbits to orbits, it is clear that the w_i form a reduced system independent of q and p. It can be computed using nothing more than the chain rule:

$$
\begin{aligned}
\dot{w}_1 &= 2q \cdot \dot{q} = 2w_1\frac{\partial H}{\partial w_2} + 4w_2\frac{\partial H}{\partial w_3},\\
\dot{w}_2 &= q \cdot \dot{p} + \dot{q} \cdot p = -2w_1\frac{\partial H}{\partial w_1} + 2w_3\frac{\partial H}{\partial w_3},\\
\dot{w}_3 &= 2p \cdot \dot{p} = -4w_2\frac{\partial H}{\partial w_1} - 2w_3\frac{\partial H}{\partial w_2}.
\end{aligned} \tag{17}
$$

Probably any student confronted with (16) and comfortable with the chain rule would hit on the idea of computing (17). However, from the general theory of symplectic reduction we know that we have carried out the reduction $\mathbb{R}^6/\mathrm{O}(3)$ and that the resulting system (17) will be Poisson.

Writing (17) explicitly as a Poisson system—that is, in the form of equation (8)—yields

$$
\dot{w} = \underbrace{\begin{pmatrix} 0 & 2w_1 & 4w_2 \\ -2w_1 & 0 & 2w_3 \\ -4w_2 & -2w_3 & 0 \end{pmatrix}}_{K(w)} \underbrace{\begin{pmatrix} \frac{\partial H}{\partial w_1} \\ \frac{\partial H}{\partial w_2} \\ \frac{\partial H}{\partial w_3} \end{pmatrix}}_{\nabla H(w)} = K(w)\nabla H(w). \tag{18}
$$

Note the analogy with the rigid body equations in Poisson form (equations (3), (8) and (9)). Only the Poisson tensor is different and the Hamiltonian is arbitrary. Thus, the reduced central force equations (18) form another and very easily accessible example of a Poisson system.

Step 2. Finding the Casimir and passing to a two-dimensional symplectic leaf. The Poisson tensor $K(w)$ in (18) is linear in w so it is natural to ask which Lie algebra it is associated with. That is (see equation (10)), we seek three matrices W_1, W_2, W_3 such that

$$[W_1, W_2] = 2W_1, \quad [W_1, W_3] = 4W_2, \quad [W_2, W_3] = 2W_3. \tag{19}$$

The reader can check that the matrices

$$W_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

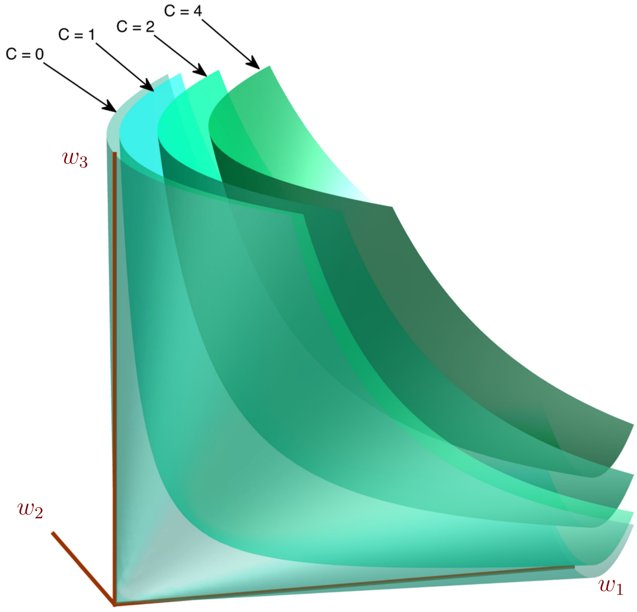

Figure 4. The reduced phase space of central force problems. Here $w_1 = \|q\|^2$, $w_2 = q \cdot p$, and $w_3 = \|p\|^2$ are the reduced coordinates, and $C = w_1 w_3 - w_2^2$ is the Casimir corresponding to angular momentum. The motion is restricted to level sets of the Casimir C. These are the symplectic leaves or symplectic reduced spaces for central force problems. There are three types of leaves: for $C > 0$ the leaves are one sheet of a two-sheeted hyperboloid (topologically a plane), and for $C = 0$ (i.e., for zero angular momentum) they are the single point at the origin and the cone meeting the origin (topologically a cylinder). All orbit types fit smoothly together.

obey equation (19). These matrices are a basis for the Lie algebra $\mathfrak{sp}(2)$ of 2×2 matrices of zero trace. We will come to the question of why this *particular* Lie algebra appears later, in section 3.3.

To draw the phase portrait of the reduced central force equations (18), we first seek a Casimir C, that is, a function such that $K(w)\nabla C \equiv 0$. The solution is

$$C(w) = w_1 w_3 - w_2^2.$$

The symplectic leaves are the level sets of C, which are hyperboloids for $C > 0$ and a cone minus its point for $C = 0$. (There is also a singular leaf consisting of the single point $(w_1, w_2, w_3) = (0, 0, 0)$.) These are shown in figure 4, giving a foliation of phase space. Note that, when written in terms of the original variables (q, p), C is the square of the total angular momentum, $C(w) = \|q \times p\|^2$—in close analogy with the rigid body. We point out, however, that the foliation here is quite different from the rigid body, consisting of hyperboloids instead of spheres.

Step 3. Obtaining the phase portrait as the level sets of the Casimir and the energy. The orbits of (18) are the intersection of the symplectic leaves $C(w) = $ const. with the energy level sets $H(w) = $ const.—again in analogy with the rigid body. This allows the phase portrait to be read off, for any O(3)-invariant Hamiltonian, for all values of the total angular momentum at once, and to easily visualise how all the orbits fit together.

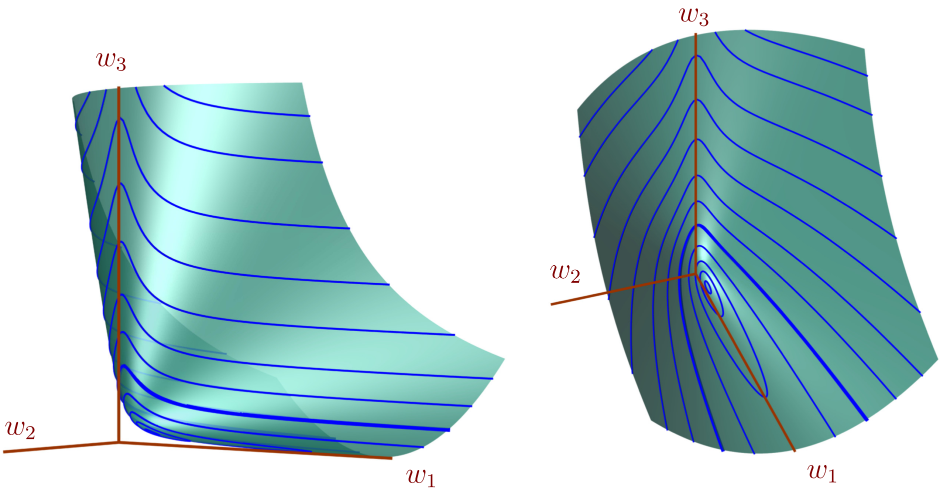

Figure 5. Two views of the phase portrait for the reduced Kepler problem, given as level sets of the reduced Hamiltonian H restricted to the coadjoint orbit $C = 0.6$. The level set marked by a thicker curve corresponds to zero energy $H = 0$. This is the *escape energy*, so orbits below it are bounded and periodic, whereas curves above it are unbounded. Notice the stable relative equilibrium at the bottom, corresponding to the circular solution of the Kepler problem where $\|q\|^2$ and $\|p\|^2$ are constant and $q \cdot p = 0$. The region $[0, 12] \times [-6, 6] \times [0, 12]$ is shown.

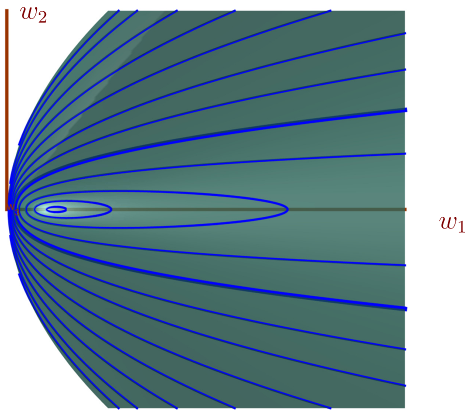

Figure 6. A more standard choice of reduced coordinates is (w_1, w_2). The phase portrait of the reduced Kepler problem for $C = 0.6$ is shown here in these coordinates. A large part of the phase portrait is squashed near the w_2-axis, and these coordinates cannot be used for zero angular momentum ($C = 0$).

We have drawn such phase portraits for three systems: (i) the Kepler problem with $\widetilde{H}(q, p) = \frac{1}{2}\|p\|^2 - 1/\|q\|$ (shown in figures 5 and 6); (ii) a Hamiltonian that exhibits homoclinic orbits, $\widetilde{H}(q, p) = \|q\|^2 + \|p\|^2 + (q \cdot p)^4 - 4(q \cdot p)^2$ (shown in figure 7); and (iii) a Hamiltonian with startlingly complex phase portrait, $H(w) = \sum_{i=1}^{3} \cos w_i$ (shown in figure 8).

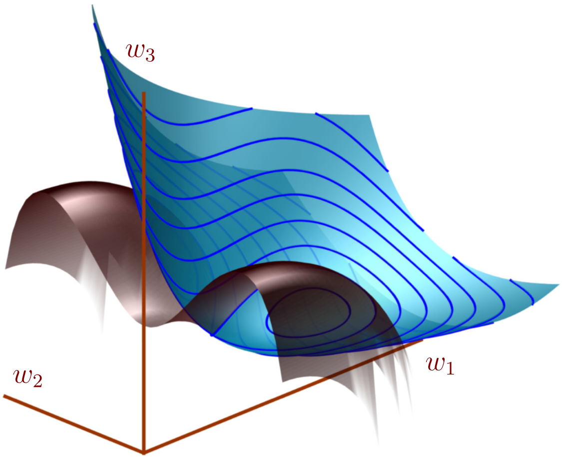

Figure 7. Two views of the phase portrait of a central force problem with homoclinic orbits. Here $H(w) = w_1^2 + w_3^2 + w_2^4 - 4w_2^2$. The energy level set $H = 2$ (shown on the left) intersects the Casimir level set $C = 1$ to create two homoclinic orbits. The situation for is similiar on other symplectic leaves: the orbit $H^{-1}(2\alpha) \cap C^{-1}(\alpha)$ is homoclinic. The region $[0, 4] \times [-2, 2] \times [0, 4]$ is shown.

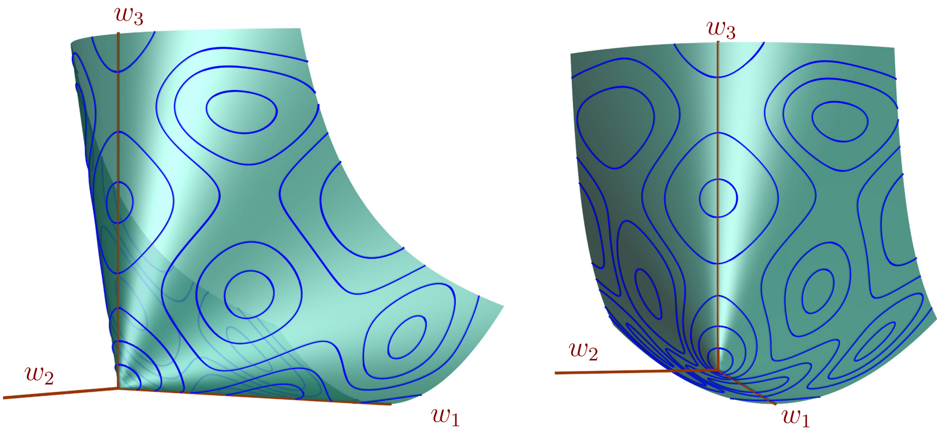

Figure 8. Two views of the phase portrait for a more complex central force problem at zero angular momentum. Here $H(w) = \sum_{i=1}^{3} \cos w_i$. The interaction between the level sets of H and of the Casimir $C = w_1 w_3 - w_2^2$ creates a complex phase portrait. The region $[0, 12] \times [-6, 6] \times [0, 12]$ is shown.

3.2. Reconstruction

Thus far we have reduced the original set of six differential equations, equation (16), to three differential equations, equation (18), and shown how the solutions of the reduced equations may be visualised as the level sets of the energy and the Casimir. Strikingly, we have not yet used the conserved quantity associated with the symmetry. In fact, we have avoided mentioning it at all. Of course it is the angular momentum, $J(q, p) = q \times p$. This will now make an entrance as we reconstruct the full motion of the system.

Suppose that the reduced equations (18) have been solved to yield a reduced orbit $w(t)$. If this $w(t)$ is substituted into the original, unreduced, system (16), it becomes a system of six linear, nonautonomous equations that reconstruct the full motion. However, a much more dramatic simplification of the full motion is possible. This is because the values of both $w(t)$ and of the angular momentum $J(q(t), p(t)) = J(q(0), p(0)) = q(0) \times p(0)$ are known on the solution curve $(q(t), p(t))$; these determine $(q(t), p(t))$ up to a single scalar function of t. This function can be found by integrating a single function of time, as follows.

Let $(\tilde{q}(t), \tilde{p}(t))$ be *any* curve in phase space that matches the known value of $w(t)$—that is, such that $(\|\tilde{q}(t)\|^2, \tilde{q}(t) \cdot \tilde{p}(t), \|\tilde{p}(t)\|^2) = w(t)$. Such a curve can be determined using only algebra. The desired solution, $(q(t), p(t))$, shares the same values of w and J. Since w_1, w_2, and w_3 are the defining invariants of the symmetry group, for each time t there must be an element of the symmetry group O(3), say A, that maps $(\tilde{q}(t), \tilde{p}(t))$ to $(q(t), p(t))$. But this map also has to preserve the angular momentum J. The O(3) action is $(q, p) \mapsto (Aq, Ap)$, so the action on angular momentum is $q \times p \mapsto (Aq) \times (Ap) = (\det A) A (q \times p)$. If the angular momentum is nonzero, the orthogonal matrices A that preserve $q \times p$ are exactly the rotations about the axis $q \times p$. (We focus on the case of nonzero angular momentum; if the angular momentum is zero, then q and p are collinear and remain pointing in the same direction forever, and are easily determined from w.) That is, the solution can be written in the form

$$\begin{aligned} q(t) &= A(t)\tilde{q}(t), \\ p(t) &= A(t)\tilde{p}(t), \end{aligned}$$

where $A(t) \in \mathrm{SO}(3)$ is some rotation about the fixed axis $q(t) \times p(t)$. Substituting this ansatz into the equations of motion (16) determines $A(t)$.

> This is an instance of a general result that the reduced motion and the conserved momentum together determine the full motion up to an element of a certain subgroup G_μ of the symmetry group G. This is the *isotropy subgroup* associated with the value μ of the momentum map J. The symmetry group G has a natural action on values of the momentum, called the coadjoint action; in this case we can write $J(q, p) = q \times p =: \mu \in \mathbb{R}^3$ and the coadjoint action is $A \cdot \mu = (\det A) A\mu$. So in this case, the group elements that fix μ are the rotations about the axis μ.

To find this rotation $A(t)$, let $w(0)$ be the initial value of $w(t)$. We can choose coordinates on $\mathbb{R}^3$ so that

$$q(0) = \begin{pmatrix} \sqrt{w_1(0)} \\ 0 \\ 0 \end{pmatrix}, \quad p(0) = \begin{pmatrix} \frac{w_2(0)}{\sqrt{w_1(0)}} \\ \sqrt{w_3(0) - \frac{w_2(0)^2}{w_1(0)}} \\ 0 \end{pmatrix}.$$

Here the coordinates have been chosen to match the given value of $w(0)$ and so that μ is aligned with the z-axis. Therefore it remains aligned with the z-axis for all time and the solution can be written

$$q(t) = A(t)\begin{pmatrix} \sqrt{w_1(t)} \\ 0 \\ 0 \end{pmatrix}, \quad p(t) = A(t)\begin{pmatrix} \frac{w_2(t)}{\sqrt{w_1(t)}} \\ \sqrt{w_3(t) - \frac{w_2(t)^2}{w_1(t)}} \\ 0 \end{pmatrix}, \tag{20}$$

where

$$A(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents the unknown rotation about the z-axis by an angle $\theta(t)$ which is to be determined. (Equivalently, we can rotate the initial condition so that it lies in the (x, y) plane with $q(0)$ lying on the positive x-axis.)

Now inserting the ansatz (20) into the equations of motion (16), and using the reduced central force equations (18) that are satisfied by $w(t)$, yields the reconstruction equation

$$\dot{\theta} = 2\frac{\partial H}{\partial w_3}\frac{\|\mu\|}{w_1}. \tag{21}$$

As the right-hand side is a known function of t, the solution to (21) is the phase

$$\theta(T) = \int_0^T 2\frac{\partial H}{\partial w_3}(w(t))\frac{\|\mu\|}{w_1(t)}\,\mathrm{d}t. \tag{22}$$

Note that the formula for the phase is not unique as it depends on the choice of coordinates in (20).

Thus we see directly that the central force problem is integrable for all H. Orbits for which $w(t)$ is a constant are called 'relative equilibria': the full motion is periodic with period $2\pi w_1/\left(2\frac{\partial H}{\partial w_3}\|\mu\|\right)$. Orbits for which $w(t)$ is periodic are called 'relative periodic orbits': the full motion is quasiperiodic with second period $2\pi/\theta(T)$, where T is the period of $w(t)$. Relative periodic orbits can be seen in figures 5, 7, and 8 as closed curves in the reduced phase space. After one period, T, of the reduced dynamics, the full motion does not return to its starting point but instead picks up a phase $\theta(T)$ (in this case, a rotation in the plane of the motion)—an example of a geometric phase, analogous to the flip in the hammer toss experiment shown in figure 2.

3.3. Dual pairs

(We remind the reader that this and forthcoming sections use some introductory concepts from geometric mechanics, found, for example, in Arnold's treatise [2].)

Now we come to the apparently surprising entrance of $\mathfrak{sp}(2)$, the 2×2 matrices of zero trace. The key to this is that in writing the Hamiltonian in the form $H(\|q\|^2, q \cdot p, \|p\|^2)$, we are being given a lot of extra information about the system compared to the default situation, in which all we know is that the Hamiltonian is invariant under a given group action. Namely, we are being given

i. the reduced Hamiltonian explicitly; and
ii. the function $(q, p) \mapsto (\|q\|^2, q \cdot p, \|p\|^2)$.

In this section we will explore the consequences of this extra information.

Let G_2 be a matrix group with a Hamiltonian action on $T^*\mathbb{R}^n$ and momentum map $J_2 := T^*\mathbb{R}^3 \to \mathfrak{g}_2^*$. Functions of the form $H \circ J_2$ are called *collective Hamiltonians* and play an important and useful role in Hamiltonian dynamics, although they seem to be much less well known than symmetric Hamiltonians and their associated tools of Noether's theorem and symplectic reduction. Collective motion is covered in section 28 of Guillemin and Sternberg [9] and section 12.4 of Marsden and Ratiu [15], for example. The key facts are that

i. J_2 is a Poisson map, i.e. $\{F, G\}_{\mathfrak{g}_2^*} \circ J_2 = \{F \circ J_2, G \circ J_2\}_{T^*\mathbb{R}^n}$ for all functions $F, G := T^*\mathbb{R}^n \to \mathbb{R}$; and
ii. the Hamiltonian vector field of $H \circ J_2$ on $T^*\mathbb{R}^n$ descends to the Hamiltonian vector field of H on $\mathfrak{g}_2^*$.

In the treatment of the central force problem above, we have verified these facts 'by hand' for this example.

In the central force problem, $J_2 = (\|q\|^2, q \cdot p, \|p\|^2)$ is the momentum map for the action of $\mathrm{Sp}(2)$ on $T^*\mathbb{R}^3$ given by

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}, \quad ad - bc = 1.$$

This can be checked by writing down the Hamiltonian vector fields of the components of J_2, which are the generators of this action, or by evaluating their Poisson brackets, e.g. $\{\|q\|^2, \|p\|^2\} = 4q \cdot p$.

Thus, the dynamics of collective Hamiltonians are exceptionally easy to reduce. The reduced system is always Lie–Poisson system and can be written down explicitly. In the central force problem, not only do we have a collective Hamiltonian, but we know even more, namely that *all* $\mathrm{O}(3)$-invariant Hamiltonians are collective.

A second reason for the appearance of $\mathfrak{sp}(2)$ is that the matrix groups

$$\begin{aligned} &G_1 \coloneqq \mathrm{O}(3), \quad A \cdot (q, p) = (Aq, Ap) \\ &G_2 \coloneqq \mathrm{Sp}(2), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (q, p) = (aq + bp, cq + dp) \end{aligned} \tag{23}$$

are *mutual centralizers* within the group $\mathrm{Sp}(6)$ of all symplectic linear maps on $T^*\mathbb{R}^3$. That is, G_2 consists of those matrices in $\mathrm{Sp}(6)$ that commute with all matrices in G_1, and vice versa. This is an example of a *dual pair*. Amongst its remarkable consequences [11, 21] are that the momentum maps J_1 and J_2 are quadratic functions that generate all polynomial invariants of G_2 and G_1, respectively; that G_1 is a symmetry group of J_2 and vice versa; and that the Hamiltonian $H \circ J_2$ is G_1-invariant (as we used above) and vice versa. The dual pair was introduced into representation theory by Roger Howe [11], who gave examples of its widespread occurrence in mathematical physics, including fundamental formalism, massless particles, classical equations (wave, Laplace, Maxwell, and Dirac) and supersymmetry.

With this in mind, we can see another motivation for the appearance of $\mathrm{Sp}(2)$, namely that it is a symmetry of the angular momentum level sets, for

$$(aq + bp) \times (cq + dp) = (ad - bc)(q \times p) = q \times p.$$

Remarkably, Howe [11] was able to classify the main dual pairs (the 'irreducible reductive' ones). There are just seven families of these, with $(G_1, G_2) = (\mathrm{GL}(n, F), \mathrm{GL}(m, F))$, where the field F is the real numbers, the complex numbers, or the quarternions; $(\mathrm{O}(p, q, F), \mathrm{Sp}(2k, F))$ (the groups preserving a Hermitian (resp. skew-Hermitian) bilinear form; this is the type that arises in the central force problem); and $(\mathrm{U}(p, q), \mathrm{U}(r, s))$.

We close this discussion of the geometry of central force problems by comparing in table 1 the relevant structures for the free rigid body and for central force problems.

Table 1. Analogous concepts for the rigid body and the cental force problem.

	Rigid body	Central force
Symmetry group	$\mathrm{SO}(3)$	$\mathrm{O}(3)$
Phase space	$\mathrm{SO}(3) \times \mathbb{R}^3$	$\mathbb{R}^3 \times \mathbb{R}^3$
Dual group	$\mathrm{SO}(3)$	$\mathrm{Sp}(2)$
Action	$A \cdot (Q, m) = (AQ, m)$	$A \cdot (q, p) = (Aq, Ap)$
Momentum	$\mu = Qm$	$\mu = q \times p$
Coadjoint action	$A\mu$	$(\det A)A\mu$
Casimir	$\Vert\mu\Vert^2 = m_1^2 + m_2^2 + m_3^2$	$\Vert\mu\Vert^2 = w_1 w_3 - w_2^2$
Reduced coordinates	m	$w = (\Vert q\Vert^2, q \cdot p, \Vert p\Vert^2)$
Reduced basis	$\begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}$	$\begin{pmatrix} -w_2 & 2w_1 \\ -2w_3 & w_2 \end{pmatrix}$
Poisson structure	$\begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2w_1 & 4w_2 \\ -2w_1 & 0 & 2w_3 \\ -4w_2 & -2w_3 & 0 \end{pmatrix}$

3.4. More bodies in more dimensions

In our discussion of the dynamics of *one* body moving in *three* dimensions under a central force, it is easy to see that the three-dimensionality of space does not play any significant role in the reduction procedure. Similarly we can guess that with more than one body, one should replace $(q \cdot q, q \cdot p, p \cdot p)$ by the dot products of all pairs of qs and ps. This is in fact the case, and the resulting reduction is an application of the Howe dual pair $(\mathrm{O}(n, \mathbb{R}), \mathrm{Sp}(2k, \mathbb{R}))$. The $\mathrm{O}(3)$-invariant motion of $k = 1$ body in $n = 3$ dimensions reduces to a Lie–Poisson system in $\mathfrak{sp}(2)$; the $\mathrm{O}(n)$-invariant motion of k bodies in any dimension n reduces to a Lie–Poisson system in $\mathfrak{sp}(2k)$. The resulting structure was explored by Sadetov in 2002 [24].

First, note that mechanical systems with pairwise interactions typically have Hamiltonians of the form

$$H(q, p) = \sum_{i=1}^{k} \frac{1}{2m_i} \Vert p_i \Vert^2 + \sum_{1 \leqslant i < j \leqslant k} V(\Vert q_i - q_j \Vert^2), \tag{24}$$

where the positions of the bodies are $q_1, \ldots, q_k$, where $q_i \in \mathbb{R}^n$ and their momenta are $p_1, \ldots, p_k$. Such systems are invariant under $\mathrm{E}(n)$, the Euclidean group of translations and rotations. The translation degrees of freedom, being commutative, are easy to eliminate using centre-of-mass coordinates. An explicit expression for canonical coordinates on the reduced phase space is provided by Jacobi–Bertrand–Haretu coordinates [15, 19, 24], having been fully worked out in Spiru Haretu's 1878 PhD thesis [10]; we will assume this has been done so that we are given an $\mathrm{O}(n)$-invariant Hamiltonian.

The action of $\mathrm{O}(n)$ on the positions and momenta of k bodies is given by

$$A \cdot (q_1, \ldots, q_k, p_1, \ldots, p_k) = (Aq_1, \ldots, Aq_k, Ap_1, \ldots, Ap_k).$$

The invariants of this action are the pairwise dot products of the positions and momenta, which we collect into a map

$$\varphi(q_1,\ldots,q_k,\,p_1,\ldots,p_k) = X = \begin{pmatrix} -M^\top & L \\ K & M \end{pmatrix},$$

where the $k \times k$ matrices M, L, and K are given by

$$M_{ij} = q_j \cdot p_i, \quad L_{ij} = q_i \cdot q_j, \quad K_{ij} = p_i \cdot p_j.$$

As we can guess from the one-body case, φ is the momentum map for the action of $\mathrm{Sp}(2k)$ on $T^*\mathbb{R}^{n\times k}$ given by

$$B \cdot (q_1,\ldots,q_k,\,p_1,\ldots,\,p_k) = [q_1,\ldots,q_k,\,p_1,\ldots,\,p_k]B.$$

All $\mathrm{O}(n)$-invariant Hamiltonians H are collective and can be written in the form $H = H \circ \varphi$. (If the original problem was of type (24), then H is already known explicitly.) Thus, their equations of motion reduce to Lie–Poisson systems

$$\dot{X} = \{X,\,H\}$$

on $\mathfrak{sp}(2k)^*$. The Lie algebra $\mathfrak{sp}(2k)$ consists of all $2k \times 2k$ matrices of the form $x := \begin{pmatrix} -\alpha^\top & \beta \\ \gamma & \alpha \end{pmatrix}$, where $\alpha,\,\beta,\,\gamma$ are $k \times k$ matrices and $\beta,\,\gamma$ are symmetric. The dual $\mathfrak{sp}(2k)^*$ can be identified with $\mathfrak{sp}(2k)$ by the inner product $\langle X,\,x\rangle = \frac{1}{2}\mathrm{tr}(X^\top x)$.

The unreduced system, on $T^*\mathbb{R}^{n\times k}$, has dimension $2nk$. The reduced system, (3.4), has phase space a vector space of dimension $\dim \mathfrak{sp}(2k)^* = k(2k+1)$. At first sight it is surprising that the dimension of the phase space has not been itself 'reduced' when $k \geqslant n$. For example, for the three-body problem (24) we have $k = 2$ and $n = 2$; the original phase space (in centre-of-mass coordinates) has dimension $2nk = 8$ and $\dim \mathfrak{sp}(4)^* = 10$. Some reduction!

The first point to note is that the map φ does not map onto all of $\mathfrak{sp}(2k)^*$. Its range is the positive semi-definite matrices of rank $\leqslant n$ times $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. (In the one-body example, this is the solid cone $w_1 w_3 - w_2^2 \geqslant 0$ in $\mathbb{R}^3$ shown in figure 4.) This submanifold has dimension $2nk - \frac{1}{2}n(n-1)$, which *is* a reduction in dimension. The 'extra variables' in which the reduced system is written in (3.4) is the price we pay for embedding the reduced system, with all its complicated topology of symplectic leaves, in a vector space. Second, the symplectic leaves themselves have still lower dimension. The Casimirs are $\mathrm{tr}X^{2j}$, $j = 1,\ldots,\lfloor n/2\rfloor$, so the top-dimensional leaves have dimension $2nk - \frac{1}{2}n(n-1) - \lfloor n/2\rfloor$. For $n = 3$, this is $6k - 4$, that is, symplectic reduction has lowered the dimension by 4.

In the classic three-body problem, the original phase space including the centre of mass has dimension 18; removing the centre of mass drops to dimension 12; the Lie–Poisson system (3.4) has dimension 10; the image of φ has dimension 9; and the top-dimensional symplectic leaves have dimension 8. There are also lower-dimensional leaves corresponding to planar motions, zero-angular-momentum motions, and motions in which some of the bodies coincide.

4. Discussion

The use of invariants to work with quotients by symmetry groups is extremely well established, being the main reason that invariants were originally introduced in the 19th century. In dynamical systems, it has become a powerful tool in bifurcation theory [8], but is less well established in geometric mechanics [17]. Perhaps one reason for this is that while it can be powerful in specific examples, as here, it is not universally applicable (orbits are not always specified by the values of invariants) and the geometry of the orbits and invariants has to be worked out in each case. Cushman and Bates [5] work out many examples of reduction in detail in this way, for example. Olver [20] in examples 6.43 and 6.50 treats the central force problem much as we have done here. Lerman, Montgomery, and Sjamaar [14] treat the case of k bodies in n-dimensions, noting the connection with dual pairs. The full description of the reduction to $\mathfrak{sp}(2k)^*$ is due to Sadetov [24] and is also covered by Dullin [6]. The brevity of Howe's list of dual pairs [11] suggests that there will not be many examples arising naturally in physics that possess the same elegant structure as the central force problem; one example is the reduced regularised Kepler n-body problem, which involves $\mathfrak{u}(m, m)^*$, where $m = n(n-1)/2$ [3].

Acknowledgments

This work has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 691070. RM was supported by the Marsden Fund of the Royal Society of New Zealand. We would like to thank Matt Wilkins and Milo Viviani for their comments.

References

[1] Arnold V I 1966 Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits *Ann. Inst. Fourier* **16** 319–61
[2] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[3] Arunasalam S, Dullin H R and Nguyen D M H 2014 The Lie–Poisson structure of the symmetry reduced regularised *n*-body problem *J. Phys. A: Math. Theor.* **48** 065202
[4] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: MacGraw-Hill)
[5] Cushman R H and Bates L M 2012 *Global Aspects of Classical Integrable Systems* (Basel: Birkhäuser)
[6] Dullin H R 2013 The Lie–Poisson structure of the reduced *n*-body problem *Nonlinearity* **26** 1565
[7] Feynman R P, Leighton R B and Sands M 1965 *The Feynman Lecctures on Physics* vol 1 (Reading, MA: Addison-Wesley)
[8] Golubitsky M I, Stewart I and Schaeffer D 1988 *Singularities and Groups in Bifurcation Theory: Vol II* (*Applied Mathematical Sciences* vol 69) (New York: Springer)
[9] Guillemin V and Sternberg S 1990 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)
[10] Haretu S 1878 *Thèses Presentèes a la Faculté des Sciences de Paris* (Paris: Gauthier-Villars)
[11] Howe R 1985 Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons *Applications of Group Theory in Physics and Mathematical Physics* (*Lectures in Applied Mathematics* vol 21) pp 179–207
[12] Khesin B and Wendt R 2009 *The Geometry of Infinite-Dimensional Groups* (Berlin: Springer)
[13] Landau L D and Lifshitz E M 1976 *Mechanics* 3rd edn (Butterworth: Heinemann)
[14] Lerman E, Montgomery R and Sjamaar R 1993 Examples of singular reduction *Symplectic Geometry* (*London Mathematical Society Lecture Note Series* vol 192) (Cambridge: Cambridge University Press) pp 127–55
[15] Marsden J E and Ratiu T 2013 *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems* (Berlin: Springer)
[16] Marsden J and Weinstein A 1974 Reduction of symplectic manifolds with symmetry *Rep. Math. Phys.* **5** 121–30
[17] Marsden J E and Weinstein A 2001 Comments on the history, theory, and applications of symplectic reduction *Quantization of Singular Symplectic Quotients* (Basel: Birkhäuser) pp 1–9
[18] Meyer K R 1973 Symmetries and integrals in mechanics *Dynamical Systems* ed M Peixoto (New York: Academic) pp 259–73

[19] Meyer K 1999 Periodic solutions of the N-body problem *Lecture Notes in Mathematics* vol 1719 (Berlin: Springer)

[20] Olver P J 2000 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)

[21] Ortega J-P and Ratiu T 2013 *Momentum Maps and Hamiltonian Reduction* (Berlin: Birkhäuser)

[22] Schreiber K U, Klügel T, Wells J P, Hurst R B and Gebauer A 2011 How to detect the Chandler and the annual wobble of the Earth with a large ring laser gyroscope *Phys. Rev. Lett.* **107** 173904

[23] Schwarz G W 1975 Smooth functions invariant under the action of a compact lie group *Topology* **14** 63–8

[24] Sadetov S T 2002 On the regular reduction of the n-dimensional problem of $N + 1$ bodies to Euler–Poincaré equations on the Lie algebra $sp(2N)$ *Regular Chaotic Dyn.* **7** 337–50

[25] Tondeur P 1962 Structure presque kählérienne naturelle sur le fibré des vecteurs covariants d'une variété riemannienne *C.R. Acad. Sci. Paris* **254** 407–8

[26] Yamaguchi Y Y and Iwai T 2001 Geometric approach to Lyapunov analysis in Hamiltonian dynamics *Phys. Rev.* E **64** 066206

[27] Weinstein A 1983 The local structure of Poisson manifolds *J. Differ. Geom.* **18** 523–57