

Lie group foliations: Dynamical systems and integrators

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Abstract

Foliate systems are those which preserve some (possibly singular) foliation of phase space, such as systems with integrals, systems with continuous symmetries, and skew product systems. We study numerical integrators which also preserve the foliation. The case in which the foliation is given by the orbits of an action of a Lie group has a particularly nice structure, which we study in detail, giving conditions under which all foliate vector fields can be written as the sum of a vector field tangent to the orbits and a vector field invariant under the group action. This allows the application of many techniques of geometric integration, including splitting methods and Lie group integrators.

1 Introduction

In the early works of Feng Kang [8], geometric integration was taken to be the approximation of flows by elements of certain subgroups of $Diff(M)$ (the group of diffeomorphisms of the phase space M)—the groups of symplectic, volume-preserving, or contact diffeomorphisms, for example. This point of view was developed further in [14] using the Cartan classification of diffeomorphism groups and is continued here by considering the so-called nonprimitive groups, those that leave a foliation of M invariant. (A preliminary announcement of some of our results appears in [15].) That is, we ask: given a vector field whose flow preserves a given foliation of M , how can we construct integrators with the

same property? Many other interesting (and difficult!) questions can be asked about this class of systems. How can the existence of an invariant foliation be detected? What are the consequences for the dynamics of the system? Even regarding the construction of integrators, the class of all foliations seems to be too large to admit a useful theory, and we are led (following the example of Lie group integrators [9,21]) to consider foliations defined by the action of a Lie group. We introduce these with an example.

Example 1 Let $M = \mathbb{R}^2$ and consider the vector field

$$\dot{x} = xy + x(1 - x^2 - y^2), \quad \dot{y} = -x^2 + y(1 - x^2 - y^2) \quad (1)$$

In polar coordinates, this becomes

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = -r \cos \theta,$$

showing that the foliation into circles $r = \text{const.}$ is invariant under the flow. (In fact, this foliation is *singular*, because the leaf through the origin, a single point, has less than maximal dimension.) A one-step integrator preserves this foliation (i.e., is ‘foliate’ if the final value of r is independent of the initial value of θ). Of course this is easy to obtain in polar coordinates, but we shall see that no standard integrator in cartesian coordinates is foliate. The leaves of this foliation are the group orbits of the standard action of $SO(2)$ on \mathbb{R}^2 (see Figure 1, top).

We show in section 3 that in many cases, vector fields whose flows preserve such a foliation have a particularly simple structure, namely, they can be written as a sum of two vector fields, one tangent to the leaves of the foliation (being the orbits of a group action), and one invariant under the group action. This representation allows the construction of various types of foliation-preserving integrators.

Definition 1 [20,22] *Let M be a manifold of dimension m . A singular foliation F of M is a partition of M into connected immersed submanifolds (the “leaves”), such that the vector fields on M tangent to the leaves are transitive on each leaf [22]. F is regular if each leaf has the same dimension. F has codimension q if the maximum dimension of the leaves of F is $m - q$. A diffeomorphism of M is foliate with respect to F if it leaves the foliation invariant, i.e., if it maps leaves to leaves. A vector field on M is foliate if its flow is foliate. The space of smooth vector fields foliate with respect to F is denoted \mathfrak{X}_F . A (one-step) integrator is foliate with respect to F if the diffeomorphisms of M corresponding to each time step are foliate. The space of smooth vector fields tangent to the leaves of F is denoted $\mathfrak{X}_{\text{tan}}$. The space of leaves (denoted M/F) is obtained by identifying the points in each leaf together with the quotient topology.*

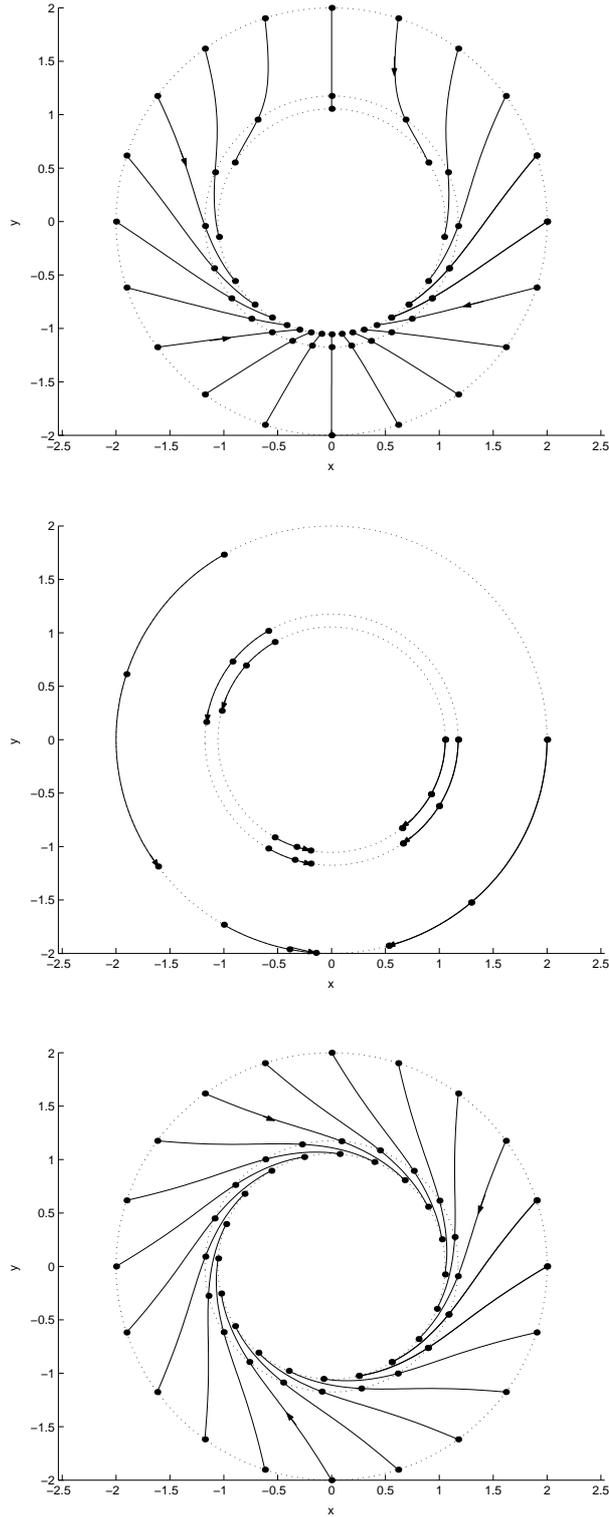


Fig. 1. Three foliate vector fields. Top: a general foliate vector field, $\dot{r} = r(1 - r^2)$, $\dot{\theta} = -r \cos \theta$ (Eq. (1)). Middle: a system with an integral, $\dot{r} = 0$, $\dot{\theta} = -r \cos \theta$. Bottom: a system with a continuous symmetry, $\dot{r} = r(1 - r^2)$, $\dot{\theta} = -(1 + r^2/5)$. All three flows map circles to circles. The dots mark times 0, 0.5, and 1.

Theorem 1 [20] \mathfrak{X}_F and $\mathfrak{X}_{\text{tan}}$ form Lie algebras. $\mathfrak{X}_{\text{tan}}$ is an ideal in \mathfrak{X}_F . A vector field X is foliate with respect to F if and only if $[X, Y] \in \mathfrak{X}_{\text{tan}}$ for all $Y \in \mathfrak{X}_{\text{tan}}$.

Theorem 2 [20] Let M and N be manifolds of dimension m and n , respectively. Let $I : M \rightarrow N$ be a smooth surjection. (If I is not onto, we replace N by $I(M)$.) Then I defines a foliation F whose leaves are given by the connected components of $I^{-1}(y)$ for each $y \in I(M)$. If I is a submersion, i.e. if dI has constant rank n , then F is a regular foliation of codimension n . In this case the space of leaves M/F is diffeomorphic to N .

Such a foliation is called *simple*. Given a vector field, one can search for simple foliations it preserves by looking for functions I such that $\dot{I} = f(I)$. These functions I might be conserved quantities in special cases. For example, the conserved momentum function J of a Hamiltonian system can evolve under $\dot{J} = cJ$ under the addition of linear friction. Then the perturbed system preserves the simple foliation defined by J [18].

For a regular simple foliation, a foliate vector field X drops to a vector field on the space of leaves M/F , which (following the tradition in geometric mechanics, [13]) we call the *reduced* system. If $\xi : [0, T] \rightarrow M/F$ is an orbit of the reduced system, the problem of finding an orbit of X is called the *reconstruction problem*. If all leaves are diffeomorphic, then reconstruction involves integrating a nonautonomous vector field on a fixed leaf. (In Example 1, the reduced system is $\dot{r} = r(1 - r^2)$; for any solution $r(t)$ of this equation, the reconstruction system is $\dot{\theta} = -r(t) \cos \theta$, a nonautonomous ODE on the leaf S^1 .)

In the application to integrators, we do not usually want to construct the reduced system explicitly since the original phase space M is usually linear and easier to work in. We would like integrators on M which preserve the foliation automatically.

Example 2 A system with k first integrals $I : M \rightarrow \mathbb{R}^k$ is foliate with respect to the level sets of the functions I . Each leaf is in fact fixed by the flow. For this reason we choose the symbol I in Theorem 2 to suggest that simple foliate systems generalize systems with first integrals. (See Figure 1, middle.)

Example 3 A system with a symmetry is foliate with respect to the orbits of the symmetry. That is, let X admit the Lie group action $\lambda : G \times M \rightarrow M$ as a symmetry, so that its flow ϕ_t is G -equivariant. Then $\lambda(g, \phi_t(x)) = \phi_t(\lambda(g, x))$, i.e., the foliation with leaves given by the group orbits $\{\lambda(g, x) : g \in G\}$ is invariant. In this case the reconstruction problem on G is easier to solve than in the general case, because it is G -invariant. (See Figure 1, bottom.)

Example 4 The Lorenz system is given by

$$\begin{aligned}\dot{x} &= \sigma y - \sigma x, \\ \dot{y} &= -y - xz - rx, \\ \dot{z} &= xy - bx.\end{aligned}$$

If $b = 2\sigma$, the system is foliate with leaves $x^2 - 2\sigma z = \text{const.}$, for

$$\frac{d}{dt}(x^2 - 2\sigma z) = -2\sigma(x^2 - 2\sigma z).$$

(A foliate integrator can be constructed as follows. We split into

$$\begin{aligned}X_1: \dot{x} &= \sigma y, \quad \dot{y} = -xz - rx, \quad \dot{z} = xy, \\ X_2: \dot{x} &= -\sigma x, \quad \dot{y} = -y, \quad \dot{z} = -2\sigma x.\end{aligned}$$

X_1 is tangent to the foliation and may be integrated using the midpoint rule, which preserves the quadratic function $x^2 - 2\sigma z$; X_2 is foliate but linear, and can be solved exactly. Composition then yields a foliate integrator.)

Example 5 A special case of the foliations defined by submersions is given by $M = N \times L$ and I is projection onto N . Each leaf is then diffeomorphic to L . (This is locally true of any simple foliation in a neighborhood in which the leaves have constant dimension.) In coordinates x on N and y on L , any foliate vector field can be written in coordinates as

$$\begin{aligned}\dot{x} &= f(x) \\ \dot{y} &= g(x, y).\end{aligned}$$

and any tangent vector field as

$$\begin{aligned}\dot{x} &= 0 \\ \dot{y} &= g(x, y).\end{aligned}$$

These foliate vector fields are also known as *skew product systems*, introduced by Anzai [1], and studied today in ergodic theory [3] and complex dynamics [10]. A special case of skew product systems is given by $N = \mathbb{R}^n$ and $L = \mathbb{R}^{m-n}$, the standard foliation on \mathbb{R}^m .

Example 6 The extension of a nonautonomous vector field on M to an autonomous vector field on $M \times \mathbb{R}$ preserves the foliation defined by $t = \text{const.}$ Most integrators are foliate and indeed, solve the reduced system $\dot{t} = 1$ exactly.

In a foliate system, one can obtain some information about part of the system (namely, the current leaf) for all time without even knowing the full initial condition. Surely this puts strong dynamical constraints on the whole system. Nevertheless, the only properties we can point to depend on the reduced or

reconstruction systems being simple in some way: having small dimension or simple (e.g. linear) dynamics.

Example 7 Burns and Wilkinson [3] study skew product systems of the form $(x, g) \mapsto (f(x), \phi(x)g)$ on $M \times G$, where G is a compact Lie group, $\phi : M \rightarrow G$, and f is measure preserving. Here the foliation is simple as in Example 5 and the systems have extra structure: the reduced system $x \mapsto f(x)$ is measure preserving, the reconstruction system $g \mapsto \phi(x)g$ is G -equivariant, and the whole system is measure preserving (with respect to the product of the measure on M with Haar measure on G). This class is a special case of the class of measure preserving systems with a symmetry. The extra structure allows them to prove that the ergodic systems are open and dense in this class.

Example 8 Consider the system on \mathbb{R}^3 with a codimension 1 foliation,

$$\dot{x} = f(x), \quad \dot{y} = g(x, y, z), \quad \dot{z} = h(x, y, z).$$

The only possible ω -limit set of the reduced system $\dot{x} = f(x)$ is a point, which suggests that in this case the ω -limit set of the whole system is either a point or a circle (periodic orbit). (If $f(x) > 0$, however, so that the reduced system has no ω -limit set, chaos is possible in the full system.) Similarly, for the system on \mathbb{R}^3 with a codimension 2 foliation,

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \quad \dot{z} = h(x, y, z).$$

the possible ω -limit sets of the reduced system in (x, y) are a points and circles, which suggests that in these cases the ω -limit set of the full system is a point, a circle, or a 2-torus. In both examples the existence of the foliation influences the possible dynamics.

2 Integrators for simple foliations

In this section we consider whether standard integrators can be foliate, and adapt geometric integrators for systems with integrals to the foliate case.

The next Theorem generalizes the fact that Runge-Kutta methods preserve arbitrary linear integrals.

Theorem 3 *Let $M = \mathbb{R}^m$ and let F be a linear foliation, i.e., a simple foliation defined by the linear function $I : \mathbb{R}^m \rightarrow \mathbb{R}^k$. Then any Runge-Kutta method is foliate.*

PROOF. Runge-Kutta methods are linearly covariant [17], hence we can apply a linear change of variables to bring I into the form $I(x_1, \dots, x_m) =$

(x_1, \dots, x_k) . Runge-Kutta methods are closed under restriction to the closed subsystem with coordinates x_1, \dots, x_k [2], hence the final values of x_1, \dots, x_k depend only on their initial values, i.e., the method is foliate.

We now give two methods for constructing foliate integrators, each based on a popular integral-preserving method.

Projection methods. Let the leaves of the foliation be the level sets of the functions $I : M \rightarrow \mathbb{R}^k$ and write the reduced system as $\dot{I} = h(I)$. Let x_n be the initial condition. Then the following algorithm provides a foliate integrator.

- Step 1. Calculate $I_n = I(x_n)$ and integrate the reduced system for one time step, giving I_{n+1} , which depends only on I_n .
- Step 2. Integrate the full vector field for one time step, giving \tilde{x} .
- Step 3. Calculate x_{n+1} by projecting \tilde{x} onto the desired leaf $I^{-1}(I_{n+1})$, e.g. by orthogonal projection.

Discrete gradient methods. [16] For simplicity we present the method for a single function I , i.e., for a codimension 1 foliation, and use a Euclidean metric. It is necessary to first split the vector field into its components tangent and orthogonal to the leaf. Since $\dot{I} = X \cdot \nabla I = h(I)$, the component of X in the direction ∇I is $(h(I)/|\nabla I|^2)\nabla I$, while the component of X tangent to the leaf can be written in the form $A(x)\nabla I$ for some antisymmetric matrix $A(x)$ [16]. That is, the full system can be written

$$\dot{x} = \left(A(x) + \frac{h(I)}{|\nabla I|^2} \right) \nabla I.$$

Extend A and h to functions \bar{A} and \bar{h} of pairs of points satisfying \bar{A} antisymmetric, $\bar{A}(x, x) = A(x)$, $\bar{h}(I, I) = h(I)$, and let $\bar{\nabla}$ be a discrete gradient. Then the discrete-gradient discretization

$$\frac{x' - x}{\tau} = \left(\bar{A}(x, x') + \frac{\bar{h}(I(x), I(x'))}{|\bar{\nabla} I(x, x')|^2} \right) \bar{\nabla} I(x, x')$$

obeys

$$I(x') - I(x) = \tau \bar{h}(I(x), I(x')),$$

so the integrator is foliate. Here τ is the time step.

One popular integral-preserving method does not generalize to foliations. The symplectic Runge-Kutta methods preserve arbitrary quadratic first integrals (and arbitrary linear symmetries) but do not preserve arbitrary quadratic foliations. To see this, consider the midpoint rule applied to a system preserving the foliation $r = \text{const.}$ as in Example 1. Let the method be $x \mapsto x'$ and write

$\bar{x} = (x + x')/2$. The method is $x' = x + \tau X(\bar{x})$. We have

$$r'^2 - r^2 = (x' + x)^t(x' - x) = 2\tau\bar{x}^t X(\bar{x}).$$

Since the vector field X is foliate,

$$\frac{d}{dt} \frac{1}{2} r^2 = x^t \dot{x} = x^t X(x)$$

is a function of r only. Therefore, $r'^2 - r^2$ is a function of $|\bar{x}|^2$ only. However, $|\bar{x}|^2$ is not in general a function of r . For example, consider the system

$$\dot{x} = -y^2 + x, \quad \dot{y} = xy + y \tag{2}$$

for which $\dot{r} = r$, $\dot{\theta} = r \sin \theta$. Applying the midpoint rule and expanding in a Taylor series gives

$$r'^2 = r^2(1 + 2\tau + 2\tau^2 + \tau^3(3 - y^2)/2) + \mathcal{O}(\tau^4)$$

which is not a function of r only.

To obtain foliate integrators which are ‘intrinsic’ in the sense that they do not involve constructing the reduced system explicitly, we have to consider special cases. For example, linear foliations can be preserved in a sense automatically, and for systems with linear symmetries, the foliation defined by the orbits of the symmetry is preserved by symmetry-preserving integrators. We therefore consider in the next section foliations defined by the orbits of a Lie group action. Not only is the foliation simple, but the foliate vector fields themselves have a nice structure which allows the construction of foliate integrators.

3 Lie group foliations

Let G be a Lie group and $\lambda: G \times M \rightarrow M$ be an action of G on M . (We write the action as $\lambda(g, x) = \lambda_g x = g \cdot x = gx$, as needed.) This group action generates a (possibly singular) foliation whose leaves are the group orbits $\lambda(G, x)$. A vector field preserving this foliation is said to be *G-foliate*. Let \mathfrak{g} be the Lie algebra of G and let \mathfrak{g}_M be the distribution tangent to the leaves, with each $\xi \in \mathfrak{g}$ associated to a vector field ξ_M on M , i.e.,

$$\mathfrak{g}_M(x) = T_x(\lambda(G, x)) = \{\xi_M(x) : \xi \in \mathfrak{g}\}.$$

(Recall that p -dimensional distribution on M is given by associating, to each point x in M , a p -dimensional subspace of the tangent space $T_x M$ [20].)

As with any foliation, any vector field tangent to the leaves is foliate. However, in the case of a Lie group foliation we have another natural class of foliate vector fields, namely those invariant under the action. For, from Theorem 1, X is foliate iff $[X, \mathfrak{g}_M] \subset \mathfrak{g}_M$; but if X is G -invariant, then $[X, \mathfrak{g}_M] = 0$. Thus, the vector field $X_{\text{tan}} + X_{\text{inv}}$ is G -foliate, where X_{tan} is tangent to the leaves and X_{inv} is G -invariant. To characterize the G -foliate vector fields we now want to turn this around and ask: When can all foliate vector fields be decomposed in this way? We shall see that this decomposition makes it easy to construct foliate vector fields explicitly and also to construct foliate integrators.

Our results can be summarized as follows. Let X be a foliate vector field. First, if G acts by isometries, then $X = X_{\text{tan}} + X_{\text{inv}}$ where X_{inv} is not only G -invariant but also perpendicular to the leaves (Theorem 5). Second, if \mathfrak{g}_M has a G -invariant complement H , then $X = X_{\text{tan}} + X_{\text{inv}}$ with $X_{\text{inv}} \in H$ (Theorem 6). This arises, for example, if the action is free and proper (Theorem 7). The decomposition is also true locally in a neighborhood of an orbit which admits a certain slice (Theorem 8). Finally, we give a counterexample to show that not all actions admit such a decomposition (Example 9).

Recall that a proper action is one for which $G \times M \rightarrow M \times M$, $(g, x) \mapsto (g.x, x)$ is proper, i.e., the inverse image of any compact set is compact. An action of a compact group must be proper. A proper action must have compact isotropy groups at all points of M .

Theorem 4 [7] *Let G be a Lie group acting properly on the manifold M . Then M has a G -invariant Riemannian metric.*

Theorem 5 *Let (M, \langle, \rangle) be a Riemannian manifold. Suppose we have a smooth group action $\lambda: G \times M \rightarrow M$ acting by isometries on the metric \langle, \rangle . Consider the foliation by the group action where the leaf through $x \in M$ is simply the orbit $\lambda(G, x)$. Then the vector field X is G -foliate if and only if the unique metric decomposition of X into tangential and perpendicular components*

$$X = X^{\parallel} + X^{\perp} \tag{3}$$

satisfies, for all $g \in G$, $\phi_g^ X^{\perp} = X^{\perp}$. In other words, X is G -foliate if and only if its component perpendicular to the leaves is G -invariant.*

PROOF. Recall that $\mathfrak{X}_{\text{tan}}$ is the space of smooth vector fields tangent to the foliation. In this case we can write

$$\mathfrak{X}_{\text{tan}}(x) = \{\xi_M(x) : \xi \in \mathfrak{g}\}. \tag{4}$$

From Theorem 1 we have that X is foliate if and only if for all $Y \in \mathfrak{X}_{\text{tan}}$,

$$[X, Y] = [X^{\parallel} + X^{\perp}, Y] \in \mathfrak{X}_{\text{tan}}. \quad (5)$$

However, $X^{\parallel} \in \mathfrak{X}_{\text{tan}}$ by definition, so X is foliate if and only if for all $Y \in \mathfrak{X}_{\text{tan}}$,

$$[X^{\perp}, Y] \in \mathfrak{X}_{\text{tan}}. \quad (6)$$

Now, suppose X is foliate. The condition above requires, in particular, that for all $\xi \in \mathfrak{g}$, $[\xi_M, X^{\perp}] \in \mathfrak{X}_{\text{tan}}$. Now, by definition of X^{\perp} we have that $\langle X^{\perp}, \xi_M \rangle = 0$. Fix $x \in M$ and $\eta \in \mathfrak{g}$. Let ϕ_t^{η} denote the flow of η_M which is given by $\phi_t^{\eta}(x) = \lambda(\exp t\eta, x)$. We then have

$$\langle X^{\perp}(\phi_t^{\eta}(x)), \xi_M(\phi_t^{\eta}(x)) \rangle = 0. \quad (7)$$

Since ϕ_t^{η} leaves the metric invariant for all t , we have

$$\langle T\phi_{-t}^{\eta} X^{\perp}(\phi_t^{\eta}(x)), T\phi_{-t}^{\eta} \xi_M(\phi_t^{\eta}(x)) \rangle = 0 \quad (8)$$

so that

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \langle T\phi_{-t}^{\eta} X^{\perp}(\phi_t^{\eta}(x)), T\phi_{-t}^{\eta} \xi_M(\phi_t^{\eta}(x)) \rangle \\ &= \langle L_{\eta_M} X^{\perp}(x), \xi_M(x) \rangle + \langle X^{\perp}(x), L_{\eta_M} \xi_M(x) \rangle. \end{aligned} \quad (9)$$

Since the action is on the left we have $L_{\eta_M} \xi_M = [\eta_M, \xi_M] = -[\eta, \xi]_M$, so that the second term in Equation (9) is 0. It follows that

$$\langle L_{\eta_M} X^{\perp}, \xi_M \rangle = 0 \quad (10)$$

which shows that $L_{\eta_M} X^{\perp}$ is perpendicular to the leaves. However, by the assumption that X is foliate we demand that $[X^{\perp}, \eta_M] = L_{\eta_M} X^{\perp} \in \mathfrak{X}_{\text{tan}}$ so that we must have $L_{\eta_M} X^{\perp} = 0$, proving G -invariance of X^{\perp} . Conversely, suppose X^{\perp} is G -invariant. It suffices to show that X^{\perp} is foliate with respect to the G orbits. But this is immediate since the flow of X^{\perp} is G -equivariant and therefore, if ϕ_t denotes the flow of X^{\perp} , $\phi_t \lambda(g, x) = \lambda(g, \phi_t(x))$ which shows that ϕ_t maps the leaf through x to the leaf through $\phi_t(x)$.

Theorem 6 *Let $\lambda: G \rightarrow M$ be a smooth action of the Lie group G on the manifold M . Suppose the tangent bundle TM admits a G -invariant splitting $TM = \mathfrak{g}_M + H$. Let the corresponding decomposition of a vector field X be $X = Y + Z$. Then X is G -foliate iff Z is G -invariant.*

PROOF. The ‘if’ part is immediate. Conversely, let X be a G -foliate vector field. We have to show that Z is G -invariant, i.e., that $[\xi_M, Z] = 0$ for all $\xi \in \mathfrak{g}$. Since X is foliate, $[\xi_M, X] = [\xi_M, Y + Z] \in \mathfrak{g}_M$; but $[\xi_M, Y] \in \mathfrak{g}_M$ since Y is tangent to the leaves by definition. Therefore $[\xi_M, Z] \in \mathfrak{g}_M$. Fix $x \in M$ and $g \in G$. Since H is G -invariant, we have

$$T_{g \cdot x} \lambda_{g^{-1}}(Z(g \cdot x)) \in H_x.$$

Letting $g = \exp(t\xi_M)$ and differentiating with respect to t at $t = 0$ gives

$$\begin{aligned} [\xi_M, Z](x) &= L_{\xi_M} Z(x) \\ &= \left. \frac{d}{dt} \right|_{t=0} \lambda_{\exp(t\xi)}^* Z(x) \\ &= \left. \frac{d}{dt} \right|_{t=0} T_{\exp(t\xi) \cdot x} \lambda_{\exp(-t\xi) \cdot x}(Z(\exp(t\xi) \cdot x)) \in H_x. \end{aligned}$$

Since the two subspaces $\mathfrak{g}_M(x)$ and H_x are complementary, we must have $[\xi_M, Z] = 0$, that is, Z is G -invariant.

The next theorem follows either from Theorem 5 or from Theorem 6. However, we also give a direct proof which constructs the decomposition explicitly relative to the choice of a connection. Recall that a free action is one in which all isotropy groups are trivial, $G_x = \{e\}$ for all $x \in M$.

Theorem 7 *Let $\lambda: G \times M \rightarrow M$ be a smooth, free, and proper action of the Lie group G on the manifold M . Let X be a G -foliate vector field. Then there exists a vector field $X_{\text{tan}} \in \mathfrak{X}_{\text{tan}}$ and a G -invariant vector field X_{inv} such that $X = X_{\text{tan}} + X_{\text{inv}}$.*

PROOF. Because the action is free and proper, $M \rightarrow M/G$ is a principal G -bundle. There is a 1–1 correspondence between the space of connections and G -invariant complements to \mathfrak{g}_M . Fix a connection \mathcal{A} and let $H_x = \ker \mathcal{A}_x$. Furthermore, there is a unique horizontal lift map $\text{hor}: T(M/G) \rightarrow TM$. It then follows that $\text{hor} \pi_* X$ is a G -invariant vector field taking values in H . Thus $X_{\text{inv}} = \text{hor} \pi_* X$ and $X_{\text{tan}} = X - X_{\text{inv}}$ provides the desired decomposition of X . X_{tan} is tangent to the foliation because $\mathcal{A}(X_{\text{tan}}) = \mathcal{A}(X - \text{hor} \pi_* X) = \mathcal{A}(X) - \mathcal{A}(\text{hor} \pi_* X) = \mathcal{A}(X)$.

Definition 2 [7] *A slice at $x_0 \in M$ of a smooth Lie group action $\lambda: G \times M \rightarrow M$ is a submanifold S of M through x_0 such that*

- (i) $T_{x_0} M = \mathfrak{g}_M(x_0) \oplus T_{x_0} S$ and for all $x \in S$, $T_x M = \mathfrak{g}_M(x) + T_x S$;
- (ii) S is G_{x_0} -invariant; and
- (iii) if $x \in S$, $g \in G$, and $\lambda(g, x) \in S$, then $g \in G_{x_0}$.

Note that $G \cdot S$ is an open, G -invariant neighborhood of the orbit $G \cdot x_0$, and every orbit in this neighborhood intersects S in a unique G_{x_0} -orbit of S . Every proper action admits a slice at every point. The dimension of an orbit which intersects S must be greater than or equal to the dimension of the orbit through x .

Unfortunately, it appears that existence of a slice is not sufficient to guarantee the existence of a G -invariant splitting of $T(G \cdot S)$. The following theorem requires two extra assumptions. However, the first is a requirement on the group itself (not on the action), while the second only concerns the action of G_{x_0} on S , reducing the dimensionality. One important special case (to be illustrated in Example 10) is when $G_x = G_{x_0}$ for all x in S , in which the second assumption is automatically satisfied with $H = TS$. The following theorem gives sufficient conditions which, together with Theorem 6, guarantee that on $G \cdot S$ every G -foliate vector field can be decomposed into tangent and invariant components.

Theorem 8 *Let $\lambda: G \times M \rightarrow M$ be a smooth action of the Lie group G on the manifold M which admits a slice S through the point $x_0 \in M$. Furthermore, assume*

- (i) \mathfrak{g} admits a G_{x_0} -invariant splitting $\mathfrak{g} = \mathfrak{g}_{x_0} \oplus \mathfrak{t}$ where the group G_{x_0} acts on \mathfrak{g} by adjoint action.
- (ii) There exists a G_{x_0} -invariant splitting $TS = (\mathfrak{g}_{x_0})_M \oplus H$ of the tangent bundle of S .

Let U be the open G -invariant neighborhood $G \cdot S$ of the orbit through x_0 . Then the tangent bundle TU admits a G -invariant splitting

$$T_u M = g \cdot H_s \oplus \left[(g \cdot \mathfrak{t})_M(u) \oplus (\mathfrak{g}_{g \cdot x_0})_M(u) \right], \quad (11)$$

where $u \in U$ is given by $u = gs$, the first distribution on the right hand side is transverse to the group orbits and the other two form a decomposition of the group directions into directions transverse and tangent to the slice respectively.

PROOF. We first check that each distribution in Eq. (11) is G -invariant. Since the action satisfies $G_{g \cdot x_0} = gG_{x_0}g^{-1}$, we have $\mathfrak{g}_{g \cdot x_0} = \text{Ad}_g \mathfrak{g}_{x_0} =: g \cdot \mathfrak{g}_{x_0}$. Therefore

$$T\lambda_g: (\mathfrak{g}_{x_0})_M(s) \simeq (g \cdot \mathfrak{g}_{x_0})_M(gs).$$

Next, since $g \cdot (\xi)_M = (g \cdot \xi)_M$ holds for any group action, we have

$$T\lambda_g: (\mathfrak{t})_M(s) \simeq (g \cdot \mathfrak{t})_M(gs).$$

It follows that $(g \cdot \mathfrak{t})_M(gs) \cap (g \cdot \mathfrak{g}_{x_0})_M(gs) = 0$. Next, the first distribution is

G -invariant by construction, and since $T\lambda_g: T_s(G \cdot s) \simeq T_{gs}(G \cdot gs)$, we must have $g \cdot H_s \cap T_{gs}(G \cdot gs) = 0$.

Finally, we must check that the splitting is well defined. Suppose $u = gs = g_1s_1$. Then without loss of generality we can take $s_1 = s$ and $g_1 = gh$ with $h \in G_{x_0}$, since by the slice property, $G_s \subset G_{x_0}$. (Each orbit in U intersects S in a unique G_{x_0} -orbit.) The splitting is then well-defined if $T\lambda_h$ maps each of the three distributions isomorphically into themselves. We clearly have that $T\lambda_h: T_sS \simeq T_{hs}S$ and that $T\lambda_h: T_s(G_{x_0} \cdot s) \simeq T_{hs}(G_{x_0} \cdot s)$, since $h \in G_{x_0}$. So what we need is that $T\lambda_h$ maps H_s into H_{hs} , which is true by assumption (ii), and that $h \cdot \mathfrak{t} = \mathfrak{t}$ for all $h \in G_{x_0}$, which is true by assumption (i).

We do not have necessary and sufficient conditions for the action to be such that the tangent and invariant vector fields span all foliate vector fields. However, the following example shows that this is not true for all group actions.

Example 9 Let $M = \mathbb{R}^2$ and $\lambda: ((a, b), (x, y)) \mapsto (x, y + a + bx)$. The orbits of λ are the lines parallel to the y -axis, so the foliate vector fields all have the form $f(x) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y}$. The tangent vector fields have the form $g(x, y) \frac{\partial}{\partial y}$. However, the vector field $X = c(x, y) \frac{\partial}{\partial x} + d(x, y) \frac{\partial}{\partial y}$ is G -invariant iff $[\xi_M, X] = [\eta_M, X] = 0$, where $\xi_M = \frac{\partial}{\partial y}$ and $\eta_M = x \frac{\partial}{\partial y}$ are the two generators of the action. This gives

$$[\xi_M, X] = \left[\frac{\partial}{\partial y}, c \frac{\partial}{\partial x} + d \frac{\partial}{\partial y} \right] = c_y \frac{\partial}{\partial x} + d_y \frac{\partial}{\partial y} = 0,$$

implying $c_y = d_y = 0$, and

$$[\eta_M, X] = \left[x \frac{\partial}{\partial y}, c \frac{\partial}{\partial x} + d \frac{\partial}{\partial y} \right] = -c \frac{\partial}{\partial x} = 0,$$

implying $c = 0$. That is, $X = d(x) \frac{\partial}{\partial y}$, and the tangent and invariant vector fields do not span the foliate vector fields.

The flow of a foliate vector field necessarily maps leaves to leaves diffeomorphically. In particular, each orbit is restricted to leaves of constant dimension. However, for vector fields of the form $X_{\text{tan}} + X_{\text{inv}}$, even more is true.

Theorem 9 *The flow of the G -foliate vector field $X = X_{\text{tan}} + X_{\text{inv}}$ preserves the isotropy subgroup of the initial condition up to conjugacy. Specifically, $G_{x(t)} = g(t)G_{x(0)}g(t)^{-1}$, where $g(t)$ satisfies Eq. (13) below.*

PROOF. We write $x(t) = \lambda(g(t), h(t))$ as in Eq. (12). The flow ϕ_t of X_{inv} does not change the isotropy subgroup. G -equivariance means $\phi_t(\lambda(g, h(0))) = \lambda(g, \phi_t(h(0)))$ for all $g \in G$, so $g \in G_{h(0)}$ iff $g \in G_{h(t)}$. Then we have $G_{x(t)} = G_{g(t)h(t)} = g(t)G_{h(t)}g(t)^{-1} = g(t)G_{h(0)}g(t)^{-1} = g(t)G_{x(0)}g(t)^{-1}$, as required.

Note that the result is not true for all G -foliate flows. In Example 9, the isotropy groups are $G_{(x,y)} = \{(a,b): a = -bx\}$, which are all invariant under conjugacy, because G is abelian. Therefore $G_{(x_1,y)}$ is not conjugate to $G_{(x_2,y)}$ for $x_1 \neq x_2$. The flow of the foliate vector field $\dot{x} = 1, \dot{y} = 0$ therefore maps leaves to leaves of non-conjugate isotropy.

4 Integrators for Lie group foliate vector fields

Given a Lie group foliate vector field $X = X_{\text{tan}} + X_{\text{inv}}$, this splitting into tangent and invariant pieces may not be unique, because it depends on the choice of G -invariant metric (or splitting of TM). In fact we have the whole family of splittings $X = (X_{\text{tan}} + Y) + (X_{\text{inv}} - Y)$ for any G -invariant vector field $Y \in \mathfrak{X}_{\text{tan}}$. Nevertheless, once such a splitting has been found, it can be used to construct foliate integrators. First, one can integrate each piece separately and compose the results: X_{tan} by any integrator for vector fields on homogeneous spaces [21], and X_{inv} by any symmetry-preserving integrator, although these are easy to construct only when the action is linear.

It is also possible to construct integrators in one piece. Write the solution

$$x(t) = \lambda(g(t), m(t)), \quad m(0) = x(0), \quad g(0) = 1, \quad (12)$$

and differentiate to get

$$\dot{x} = D_1\lambda(g, m)\dot{g} + D_2\lambda(g, m)\dot{m}.$$

Here $D_1\lambda(g, m): T_gG \rightarrow T_xM$ is the derivative of λ in its first slot, and $D_2\lambda(g, m): T_mM \rightarrow T_xM$ is the derivative of λ in its second slot. Note that the first term on the right is tangent to the foliation. Therefore, we choose \dot{g} and \dot{m} so that the first term is X_{tan} and the second term is X_{inv} :

$$\dot{g} = (D_1\lambda(g, m))^{-1}X_{\text{tan}}(\lambda(g, m)) \quad (13)$$

$$\dot{m} = (D_2\lambda(g, m))^{-1}X_{\text{inv}}(\lambda(g, m)) = X_{\text{inv}}(m). \quad (14)$$

Here we are using that the range of $D_1\lambda(g, m)$ is $\mathfrak{X}_{\text{tan}}(x)$. If the action is not free then this map is not injective—many Lie algebra elements generate the same tangent vector $X_{\text{tan}}(x)$ —and some choice of the inverse $(D_1\lambda(g, m))^{-1}$ must be made to specify a vector field on G . However, in the examples below a natural choice can be made. Thus we have extended the vector field on M to a vector field on $G \times M$ with extra foliate structure of its own. Integrating the reduced vector field $X_{\text{inv}}(m)$ by a symmetry-preserving method leaves a reconstruction problem on G . However, the vector field on G is now known

only at the time steps of the solution of $X_{\text{inv}}(m)$ which could make it difficult to achieve high orders. It is better to integrate the full system together as follows.

Theorem 10 *When $M = \mathbb{R}^n$ and the group action is linear, the following algorithm provides a foliate integrator of order p for $X = X_{\text{tan}} + X_{\text{inv}}$. (i) Let $x(t) = \lambda(g(t), m(t))$, $g(0) = 1$, $m(0) = x(0)$ and choose a differential equation on G according to Eq. (13). (ii) Apply a Runge-Kutta method of order p to $\dot{m} = X_{\text{inv}}(m)$, and the associated Runge-Kutta–Munthe-Kaas (RKMK) [21] method to the chosen differential equation on G , with stage values for m given by the corresponding stage values of the RK method applied to $\dot{m} = X_{\text{inv}}(m)$.*

PROOF. Since the group action is linear, X_{inv} has a linear symmetry and this is preserved by the Runge-Kutta method. That is, the integrator on m is G -equivariant and hence preserves the foliation into G -orbits. Applying RKMK to the differential equation on G is equivalent to applying the same RKMK method to the full system on $G \times M$ with respect to the group action of $G \times M$ on itself given by

$$\lambda((g_1, m_1), (g_2, m_2)) = (g_1 g_2, m_1 + m_2)$$

and hence it has order p . Reconstructing the solution by acting on $m(t)$ with $g(t)$ only moves the point around on the new leaf, so the total integrator is foliate.

Example 10 The matrix differential equation $\dot{L} = [A(L), L]$ motivated much of the work in Lie group integrators [9,4]. Let $G \subset GL(n)$ be a matrix Lie group, let $M = \mathfrak{g}$, the Lie algebra of G , and let G_1 be a subgroup of G which acts on M by adjoint action, i.e. $\lambda(U, L) = ULU^{-1}$ for $U \in G_1$, $L \in \mathfrak{g}$. Recall that the “isospectral manifolds” of $\mathfrak{gl}(n)$ are the sets of matrices similar by an element of $GL(n)$. In the present example, the leaves of the foliation are the sets of matrices in \mathfrak{g} which are similar by an element of G_1 , and hence are submanifolds of the isospectral manifolds.

All vector fields tangent to the leaves can be written in the form $[A(L), L]$, where $A: \mathfrak{g} \rightarrow \mathfrak{g}_1$. From Theorem 5, when \mathfrak{g} admits an adjoint invariant metric (e.g. when it is compact), all foliate vector fields can be written in the form

$$\dot{L} = [A, L] + f(L), \quad f(ULU^{-1}) = Uf(L)U^{-1} \quad \forall U \in G_1 \quad (15)$$

for some (G_1) -adjoint-invariant function $f: \mathfrak{g} \rightarrow \mathfrak{g}$.

It is not so easy to explicitly construct all invariant vector fields. The classical method requires determining a complete and independent set of differential

invariants of the action. Another approach which can work if the action is understood well enough is to choose a reference point on each orbit, choose an isotropy-invariant tangent vector at each reference point, and push them around the leaves using the group action.

For example, consider the case $G = G_1 = GL(n)$. The orbits are the conjugacy classes of matrices so a reference point on each orbit is given by the Jordan matrices. Restricting for simplicity to the diagonalizable matrices, whose isotropy groups are the group of nonsingular diagonal matrices, we first choose $f(\Lambda)$ diagonal (so that it is isotropy invariant), and then define

$$f(L) = Ug(\Lambda)U^{-1}$$

where $L = U\Lambda U^{-1}$ is the diagonalization of L . This constructs all invariant vector fields on the diagonalizable matrices.

Furthermore, this case provides a nice example of Theorem 8 in a case in which the action is not proper. The set of diagonal matrices form a slice at any diagonal matrix, on which all isotropy subgroups are equal, so assumption (ii) of the theorem is automatically satisfied. The G_{x_0} -invariant splitting of \mathfrak{g} (assumption (i)) is provided by splitting into diagonal and off-diagonal parts. Hence from Theorem 8, Eq. (15) spans all foliate vector fields in this case. We can think of the foliate vector fields as those for which the evolution of the eigenvalues of L depends only on those eigenvalues, and not on the individual entries of L .

More explicit examples of invariant vector fields are given by

$$f(L) = p(L)g(\text{tr } L, \text{tr } L^2, \dots, \text{tr } L^n),$$

where p is a real analytic function (which, when extended to matrices, maps \mathfrak{g} into \mathfrak{g}) and $g: \mathbb{R}^n \rightarrow \mathbb{R}$.

To construct foliate integrators, following Eq. (12) we represent the solution as $L(t) = U(t)F(t)U(t)^{-1}$, $U(t) \in G_1$, $F(t) \in \mathfrak{g}$, and work with the system on $G_1 \times \mathfrak{g}$,

$$\begin{aligned} \dot{U} &= A(UFU^{-1})U, & U(0) &= I, \\ \dot{F} &= f(F), & F(0) &= L(0), \end{aligned}$$

Since the symmetry of the second equation is linear, it is preserved by any linearly covariant method, such as Runge-Kutta. Runge-Kutta–Munthe-Kaas applied to the whole system provides a foliate integrator. The simplest example is the following first-order ‘‘Lie-Euler’’ method. Noting that we can take $U_n = I$, $F_n = L_n$ at each time step (only calculating the updates which move L around on the leaves), we get the method

$$U_{n+1} = \exp(\tau A(F_n)), \quad F_{n+1} = F_n + \tau f(F_n),$$

or, in terms of L ,

$$L_{n+1} = \exp(\tau A(L_n))(L_n + \tau f(L_n)) \exp(-\tau A(L_n)).$$

Example 11 Let $M = \mathbb{R}^{n \times p}$ and let the matrix Lie group $G \subset GL(n)$ act on M by left multiplication, $\lambda(U, A) = UA$ for $U \in G$, $A \in M$. As in the previous example, in the case that the action is proper (e.g. when G is compact) we can write the foliate vector fields in the form

$$\dot{A} = g(A)A + f(A)$$

for some functions $g: M \rightarrow \mathfrak{g}$ and $f: A \rightarrow A$, where f is invariant, i.e. $f(UA) = Uf(A)$ for all $U \in G$. As before it is difficult to find all invariant vector fields. Examples are $f(A) = AV(A^T A)$ for any $V: \mathbb{R}^{p \times p} \rightarrow \mathbb{R}^{p \times p}$, for $G = SO(n)$, and $f(A) = AV(\det A)$ for any $V: \mathbb{R} \rightarrow \mathbb{R}^{p \times p}$ for $G = SL(n)$. The planar systems of Eqs. (1,2) are examples with $n = 2$, $p = 1$, and $G = SO(2)$. For $G = SO(n)$, the leaves are the sets $A^T A = \text{const.}$, one of which is the Stiefel manifold $A^T A = I$.

To construct foliate integrators, following Eq. (12) we represent the solution as $A(t) = U(t)F(t)$, $U(t) \in G$, $F(t) \in \mathbb{R}^{n \times p}$, and work with the system

$$\begin{aligned} \dot{U} &= g(UF)U, & U(0) &= I, \\ \dot{F} &= f(F), & F(0) &= A(0), \end{aligned}$$

As in the previous example, RKMK applied to the whole system provides a foliate integrator. The Lie–Euler method is

$$A_{n+1} = \exp(\tau g(A_n))(A_n + \tau f(A_n)) \tag{16}$$

An example of this method is shown in Figure 2.

5 Conclusions

The situation considered here, of vector fields preserving a given foliation, can be extended in several ways. First, a system may preserve several different foliations. This is formalized in the ‘multifoliate’ structure introduced by Kodaira and Spencer [12]: the distributions form a lattice, closed under intersections and (an appropriate) join. A simple example on \mathbb{R}^3 is provided by the two distributions $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, for which the multifoliate vector fields have the form

$$\dot{x} = f(x, z), \quad \dot{y} = g(y, z), \quad \dot{z} = h(z).$$

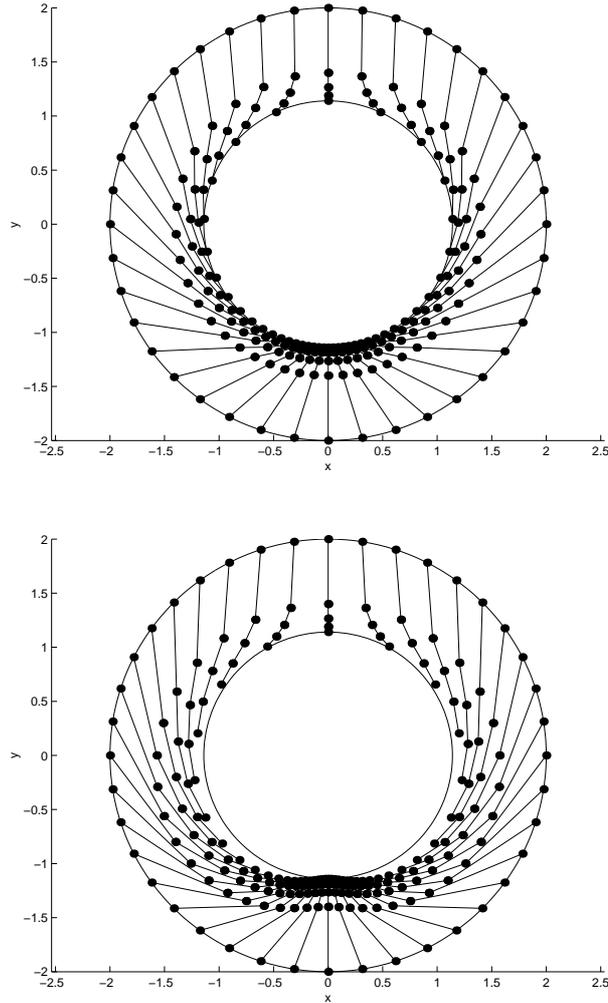


Fig. 2. Foliate vs. nonfoliate integrators. The ODE of Eq. (1) is integrated by the (foliate) Lie-Euler method Eq. (16) (top), and the (nonfoliate) Euler method (bottom). The 20 initial conditions lie on a circle of radius 2, and four time steps of 0.1 are shown. In the nonfoliate integrator, the final values do not lie on the reference circle shown. Note that the two methods coincide on $x = 0$, where the tangential component vanishes.

A more sophisticated case occurs in Hamiltonian systems with symmetry, which preserve the foliations into group orbits and into momentum level sets, and in Poisson and conformal Poisson systems, which preserve in addition the foliation into symplectic leaves [18]. We plan to study such systems more in the future [19].

Second, a foliate system may have extra structure, corresponding to the different infinite dimensional Lie subalgebras of the Lie algebra of foliate vector fields. These so-called nonprimitive Lie subalgebras have not been classified. However, many examples can be constructed by considering (i) the vector field to lie in some other Lie algebra, as of Hamiltonian or volume-preserving vector

fields; (ii) the reduced system on the space of leaves to lie in some other Lie algebra (in this case the foliated phase space has *transverse* structure); and (iii) the reconstruction system, considered as a nonautonomous vector field on a leaf, to lie in some other Lie algebra. (It is the relationships between the reduced and reconstruction systems which have not been classified.) Cartan [5] provides an interesting list of all infinite-dimensional Lie pseudogroups in two dimensions. For example, in the group

$$(x, y) \mapsto \left(\frac{xf(y) + \phi(y)}{x\psi(y) + 1}, g(y) \right),$$

the foliation $y = \text{const.}$ is preserved, and the reconstruction dynamics lie in the fractional linear group. But in the group

$$(x, y) \mapsto \left(xf'(y)^{-2} - f'''(y)f'(y)^{-2} + \frac{3}{2}f''(y)^2f'(y)^{-4}, f(y) \right),$$

what structure is preserved? In all of these cases it makes sense to ask how the structure affects the dynamics (e.g., what are its homeomorphism invariants), how it can be detected in a given system, and how integrators that lie in the corresponding group of diffeomorphisms can be constructed. From this point of view the results presented here are just a beginning.

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