A Gallery of Constant-Negative-Curvature Surfaces

Robert McLachlan

To study surfaces one must first choose how to represent them. Unfortunately, there is no canonical representation; instead, there is a long catalog of possibilities, each useful in different circumstances. Writing the surface as a graph [e.g., $z = z(x, y)$] leads to messy formulae for intrinsic quantities such as curvature; leaving the coordinates completely arbitrary includes redundant information. But when dealing with surfaces of constant negative curvature, one coordinate system stands out as being particularly apt. It goes by the name of *parametrization by Chebyshev nets*.

Actually, any surface can be locally covered by a Chebyshev net. Before I describe this coordinate system, a quick review of the geometry of surfaces is called for. Although this subject used to be part of every mathematics education, and before that, one of the centerpieces of mathematics, it is not widely taught today. In my case I found myself in the thick of the Einstein tensor and the symplectic 2-form without having laid eyes on any of the equations that go by Gauss's name.

Take arbitrary coordinates x and y on the surface, which occupies the points $\mathbf{r}(x, y) \in \mathbb{R}^3$. Define two tangent vectors and one normal vector:

$$\boldsymbol{\tau}_\mu = \mathbf{r}_{,\mu}, \qquad \mathbf{n} = \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2|}, \tag{1}$$

where Greek indices range over $\mu = 1, 2$ and the subscripts ",1" and ",2" denote $\partial/\partial x$ and $\partial/\partial y$, respectively. (The notation follows Spivak [17], Chap. 2.) Note that $\mathbf{n}$ is a unit vector, but the $\boldsymbol{\tau}_\mu$ are not necessarily unit vectors. We have the metric or first fundamental tensor $g_{\mu\nu}$ and the second fundamental tensor $h_{\mu\nu}$:

$$g_{\mu\nu} = \boldsymbol{\tau}_\mu \cdot \boldsymbol{\tau}_\nu, \qquad h_{\mu\nu} = \mathbf{n} \cdot \boldsymbol{\tau}_{\mu,\nu} = \mathbf{n} \cdot \mathbf{r}_{,\mu\nu} \tag{2}$$

As one moves along the surface, the tangent and normal vectors change according to the Gauss–Weingarten equations,

$$\begin{aligned} \boldsymbol{\tau}_{\mu,\nu} &= \boldsymbol{\tau}_\lambda \Gamma^\lambda_{\mu\nu} + \mathbf{n} h_{\mu\nu}, \\ \mathbf{n}_{,\nu} &= -\boldsymbol{\tau}_\lambda g^{\lambda\mu} h_{\mu\nu}, \end{aligned} \tag{3}$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\gamma} \cdot (g_{\mu\gamma,\nu} + g_{\nu\gamma,\mu} - g_{\mu\nu,\gamma})$, $g^{\mu\nu} = (g_{\mu\nu})^{-1}$, and repeated indices are summed on.

The six quantities in (2) are not all independent. They are related by three consistency conditions for the PDEs (3) — the Gauss–Codazzi equations:

$$K(\det g)^2 = \begin{vmatrix} -\frac{1}{2}g_{11,yy} + g_{12,xy} - \frac{1}{2}g_{22,xx} & \frac{1}{2}g_{11,x} & g_{12,x} - \frac{1}{2}g_{11,y} \\ g_{12,y} - \frac{1}{2}g_{22,x} & g_{11} & g_{12} \\ \frac{1}{2}g_{22,x} & g_{12} & g_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}g_{11,y} & \frac{1}{2}g_{22,x} \\ \frac{1}{2}g_{11,y} & g_{11} & g_{12} \\ \frac{1}{2}g_{22,x} & g_{12} & g_{22} \end{vmatrix}, \tag{4}$$

$$h_{\nu\lambda,\mu} - \Gamma^\gamma_{\nu\mu} h_{\gamma\lambda} = h_{\nu\mu,\lambda} - \Gamma^\gamma_{\nu\lambda} h_{\gamma\mu}, \tag{5}$$

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where (5) gives independent information for $(\nu, \lambda, \mu) = (1, 1, 2)$ or $(2, 2, 1)$ only. Of the three remaining degrees of freedom, two are due to the arbitrary coordinates, leaving one — which is expected because the surface could be written, e.g., $\mathbf{r} = (x, y, z(x, y))$.

The principal curvatures κ_1, κ_2 of the surface at a point are the eigenvalues of the matrix $g^{\mu\lambda}h_{\lambda\nu}$ there; their product $\kappa_1\kappa_2 = \det h/\det g$ is the Gaussian curvature K.

To construct a Chebyshev net physically, take a piece of nonstretch fabric that is loosely woven so that the angle between the threads can change. Now drape it over the surface so that the warp and weft of the fabric become coordinate lines on the surface. Because the threads cannot stretch, all coordinate lines are still parametrized by arclength — $g_{11} = g_{22} = 1$ — but g_{12} is arbitrary. The metric, therefore, takes the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}, \tag{6}$$

where $\omega(x, y)$ is the angle between two coordinate lines. Perhaps it should be called a Chebyshev fishing net to emphasize that the knots don't move. Such a coordinate system can always be constructed locally, starting from *any* two intersecting curves [16], p. 202.

The classic reference for the introduction of Chebyshev nets is [18], which is translated in full on page 37. It sounds like M. Tchébichef gave a good seminar, but it's a pity that this original treatment was never published in more detail. His rubber ball may have looked something like the one in Figure 1; in the spirit of the 19th century, I omit the details.

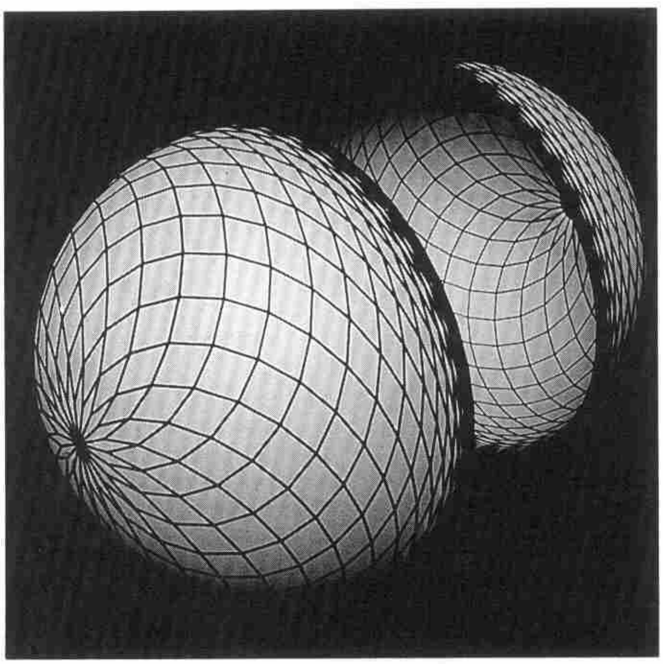


Figure 1.

Equation (4) may be written in a clumsy form, but it expresses Gauss's *theorema egregium* ("remarkable theorem"): the Gaussian curvature $K = \det h/\det g$, although apparently depending on both g and h, is, in fact, an intrinsic property of a surface; that is, it depends only on the metric g. Substituting the metric (6) into (4) gives the pleasingly simple formula

$$\frac{\partial^2\omega}{\partial x\,\partial y} = -K \sin\omega. \tag{7}$$

When $K = -1$, as for a surface of constant negative curvature, this is the famous sine–Gordon equation. In this context, it apparently appeared for the first time in the work of Hazzidakis [7]; he commented that it has solutions of the form $\omega = \varphi(x + y)$ (see the pseudosphere, p. 34) but did not write them down.

An *asymptotic direction* v^μ is one satisfying $v^\mu h_{\mu\nu} v^\nu = 0$. If the Gaussian curvature K, and hence $\det h$, are negative, then this equation has two solutions v^μ at each point. Integrating gives two families of *asymptotic lines*, each tangent to an asymptotic direction everywhere. When these are taken as coordinate lines, $h_{11} = h_{22} = 0$. Asymptotic lines have other nice properties: unless straight, they have normals tangent to the surface, and their torsion is $\sqrt{-K}$, and hence constant when K is constant ([16], p. 100).

Now for the connection: *on a surface of constant negative curvature, the asymptotic lines form a Chebyshev net.* More precisely, one can choose to parametrize any two intersecting asymptotic lines by arclength; a calculation [17], p. 365 then shows that they all are. Now solving (5) gives $h_{12} = \sqrt{-K}\,\sin\omega$, so

$$h = \begin{pmatrix} 0 & \sqrt{-K}\,\sin\omega \\ \sqrt{-K}\,\sin\omega & 0 \end{pmatrix}. \tag{8}$$

The number of functions specifying the surface has been reduced to the minimum possible, namely, one. Given consistent $g_{\mu\nu}$ and $h_{\mu\nu}$ a unique surface may be reconstructed [16], p. 146; so surfaces of constant negative curvature are locally in a 1–1 correspondence with solutions of the sine–Gordon equation.

Unfortunately this correspondence really is only local. A drawback of Chebyshev nets is that they sometimes can't be extended indefinitely over a surface, due to Hazzidakis's formula, a special case of the Gauss–Bonnet theorem [7]. Consider a coordinate rectangle $X = [x_1, x_2] \times [y_1, y_2]$ corresponding to a piece of the surface $\mathbf{r}(X)$. Then

$$\iint_X \frac{\partial^2\omega}{\partial x\,\partial y}\,dx\,dy = -\iint_X K \sin\omega\,dx\,dy$$
$$= -\iint_{\mathbf{r}(X)} K\,dA \equiv -K_T.$$

Evaluating the left-hand side gives

$$-K_T = \omega(x_1, y_1) - \omega(x_2, y_1) + \omega(x_2, y_2) - \omega(x_1, y_2).$$

For a well-defined net, the angles ω satisfy $0 < \omega < \pi$, so the total curvature K_T of the piece $\mathbf{r}(X)$ must be less than 2π in magnitude. [When $K = -1$, the area of $\mathbf{r}(X)$ must be less than 2π.] Stoker ([16], p. 199) comments that "tailors have learned this fact from experience." However, the second example below is of an infinite surface covered by a single Chebyshev net, so perhaps tailors' experience is not sufficiently broad.

The sphere has total curvature 4π, and, sure enough, can be covered by exactly two Chebyshev nets, each covering a hemisphere.

The study of surfaces of constant negative curvature was inaugurated by Ferdinand Minding in a paper published in Crelle's Journal in 1839. The content of the paper is in fact much broader, but it provides in particular the first set of concrete examples of surfaces of negative curvature, including the pseudosphere and the helical surface given below in (9). These can be found directly by making the *ansatz* of helical shape. But the earliest derivation of nontrivial solutions of sine–Gordon (one and two solitons, periodic solutions in elliptic functions, and some wave packets) is that of Seeger, *et al.* [13] in 1953.

Perhaps the reason that so many exact solutions were able to be found is that the sine–Gordon equation is completely integrable. Such integrable soliton equations have an extremely rich mathematical structure and can be solved by the method of inverse scattering [2]; this was first done for the sine–Gordon equation by Ablowitz *et al.* in 1973 [1]. Segur [14] has recently written an informal review of the practical importance of integrability. Later on, I'll give a plausible reason for just *why* such a remarkable equation should crop up in differential geometry.

Now that so many exact solutions of the sine–Gordon equation are around, modern computer tools make it a pleasure to while away a few hours and see to which surfaces they correspond [3]. Of course they can't be complete, or embed in $\mathbb{R}^3$, but if we don't mind a few cusps or self-intersections, then we're in business. It's a famous result of Hilbert [8] that complete surfaces of constant negative curvature can't embed in $\mathbb{R}^3$. In fact, singularities will always form in a surface corresponding to a single smooth solution of sine–Gordon: you can see trouble coming, because when $\omega = n\pi$ the coordinate lines are tangent to one another. Thus, a line in the coordinate plane along which $\omega = n\pi$ corresponds to a curve along which the surface has only one asymptotic line, instead of the two implied by negative curvature; in fact, the surface has only one tangent vector there. If you think of the model of sine–Gordon as the continuous limit of a line of pendula coupled by springs, it's clear that there is no solution of sine–Gordon on $\mathbb{R}^2$ that never takes any of the values $n\pi$. The principal curvatures of the surface are $\cot\omega \pm \csc\omega$, so that as $\omega \to n\pi$, they tend to zero and infinity. The pseudosphere (the first example below) has a $y = \pm x^{3/2}$-type cusp at the singularity.

However, it turns out that the singularity at $\omega = n\pi$ in the Gauss–Weingarten equations (3) is removable, so their solution exists everywhere. The coordinate lines are smooth everywhere and just blast through their points of tangency. So although it would be possible to piece together *different* surfaces smoothly along the singular lines, there is a natural continuation through them, which I've used here. As for the self-intersections, "with the sine–Gordon equation, you're living on miracle street" [15], so they might not be too bad.

The surface corresponding to a given ω may be found by integrating the Gauss–Weingarten equations. In the present case, from (3), (6), and (8), they are

$$\frac{\partial}{\partial x}\begin{pmatrix}\tau_1\\ \tau_2\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix} w_x \cot w & -w_x \csc w & 0\\ 0 & 0 & \sin w\\ \cot w & -\csc w & 0\end{pmatrix}\begin{pmatrix}\tau_1\\ \tau_2\\ \mathbf{n}\end{pmatrix},$$

$$\frac{\partial}{\partial y}\begin{pmatrix}\tau_1\\ \tau_2\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix} 0 & 0 & \sin w\\ -w_y \csc w & w_y \cot w & 0\\ -\csc w & \cot w & 0\end{pmatrix}\begin{pmatrix}\tau_1\\ \tau_2\\ \mathbf{n}\end{pmatrix}.$$

In the figures that follow I have integrated these equations numerically, and then integrated (1) to find the surface. The only special care required during the integration is to project the tangent vectors so that $\tau_1 \cdot \tau_2 = \omega$, $\tau_1 \cdot \mathbf{n} = \tau_2 \cdot \mathbf{n} = 0$, and $|\tau_1| = |\tau_2| = |\mathbf{n}| = 1$ *exactly*; this ensures that the singularities at $w = n\pi$ are removable and can be integrated through.

You may note that the surfaces all look fairly similar locally. In fact, they are all isometric — this is Minding's theorem of 1839. But they're not identical. A surface of constant negative curvature can be analytically immersed in Hilbert space such that neighborhoods of any two points are congruent (as is true for a plane, sphere, or cylinder in $\mathbb{R}^3$); but this cannot be done, even with a C^0 immersion, in any finite-dimensional Euclidean space [9].

Here are some special solutions of the sine–Gordon equation, and the constant-negative-curvature surfaces to which they correspond:

1. The Pseudosphere. Instead of the general approach using Chebyshev nets, one can look directly for those surfaces of revolution that have constant negative curvature. A special case is the pseudosphere (Fig. 2), which is $y = \sqrt{1-x^2} - \cosh^{-1}(1/x)$ rotated around the y-axis [17], p. 239. Eisenhart [4] has an early illustration. Interestingly, its area is 2π and its surface area is constant in x, that is, $dA = 2\pi\, dx$ (try this out on your calculus students — or the inverse problem!).

In the Chebyshev net parametrization, the pseudosphere corresponds to an X-independent solution of the sine–Gordon equation in the form $\partial^2\omega/\partial T^2 - \partial^2\omega/\partial X^2 = \sin\omega$ (here $X = x - y$, $T = x + y$); in fact, the homoclinic orbit connecting $\omega = 0$ and $\omega = 2\pi$ is

$$\omega = 4\tan^{-1} e^{x+y}.$$

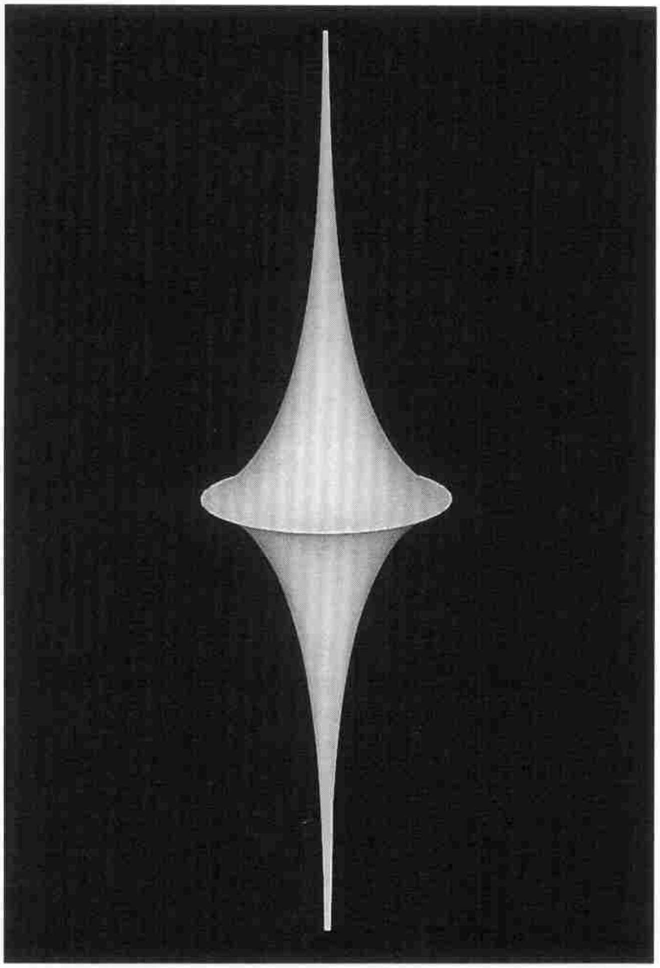


Figure 2.

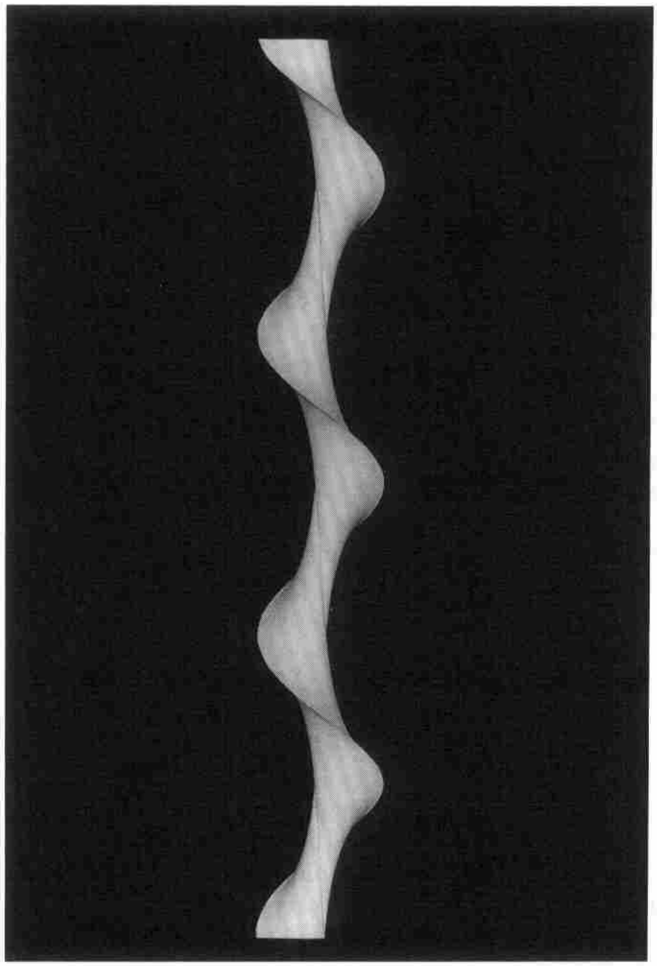


Figure 4.

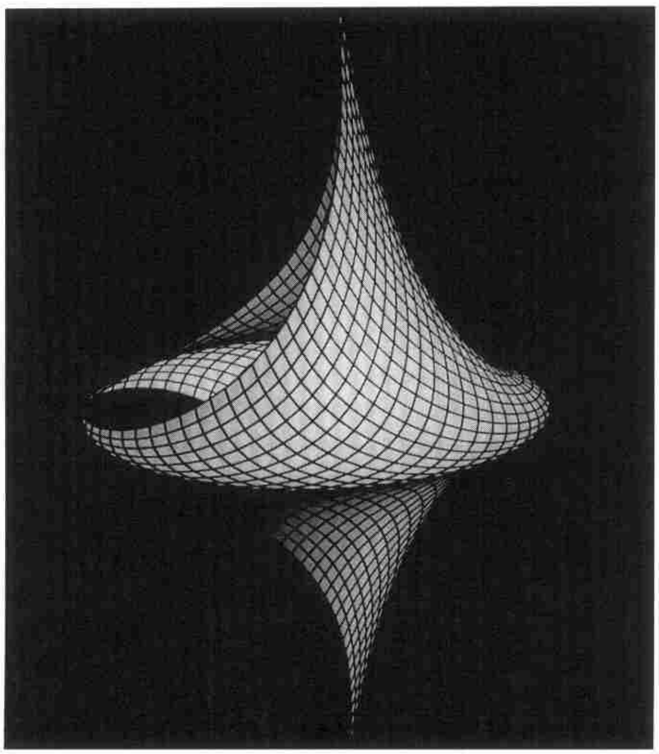


Figure 3.

The coordinate lines are shown in Figure 3; one can see how they *are* parametrized by arclength and how their angle traces out the homoclinic solution of the ($\partial^2\omega/\partial T^2 = \sin\omega$) pendulum, passing through π at the cusp. The "⅄" points correspond to $\omega \to 0$ and $\omega \to 2\pi$, i.e., $x + y \to \pm\infty$. In this limit the two sets of coordinate lines are asymptotically tangent to one another, but the singularity is never reached; because (from the Gauss–Weingarten equations) τ_{1x}, τ_{1y}, τ_{2x}, and $\tau_{2y} \to 0$, the spike must tend to infinity. If you squint a bit, you might even believe that the coordinate lines' normals lie in the surface.

2. 1-Soliton Traveling Wave. The pseudosphere solution is only a special case of the 1-soliton solution of sine–Gordon:

$$\omega = 4\tan^{-1} e^{\xi}, \qquad \xi = \eta x + y/\eta.$$

As η changes from 1, the cusp no longer closes on itself and becomes a helix. You can see the solitary wave traveling up the spiral here for $\eta = 2$ (Fig. 4). There is still a self-intersection on the central axis; the part hidden from view is like the long spike on the pseudosphere. At first sight this spiral violates the bounded area of coordinate rectangles mentioned earlier; but since the cusp lies on $\xi = 0$, a coordinate rectangle not intersecting this line does indeed have area $\leq 2\pi$. To get infinite area you need a diagonal strip $0 < \xi < c$ in the coordinate plane.

By assuming a helical shape this surface can be given explicitly as

$$\mathbf{r}(x, y) = (-2\alpha \operatorname{sech}\xi \sin \nu\xi, 2\alpha \operatorname{sech}\xi \cos \nu\xi, \quad x - 2\alpha \tanh\xi), \tag{9}$$

where $\nu = 1/\eta$ and $\alpha = 2/(\eta + 1/\eta)$ [6]. Lamb [10], while studying the motion of curves, found the sine–Gordon equation as a possible evolution equation for curves of constant curvature (κ) or of constant torsion; the latter corresponds to one of our coordinate lines, which turns out to be moving normal to itself with velocity $\sin(\int \kappa\, dx)$. The curve then sweeps out a surface of constant negative curvature, such as the one in Figure 4.

This point of view provides an explanation of why a special equation such as sine–Gordon occurs in the theory of surfaces. The sine–Gordon equation is one of the AKNS hierarchy of integrable partial differential equations [2], and it has been shown in [12] that the Serret–Frenet equations (the analogue for space curves of the Gauss–Weingarten equations), are equivalent to a special case of the AKNS scattering problem. Thus, local time-evolution equations can be given to the curve, which result in its curvature and torsion — and hence the properties of the surface it sweeps out — satisfying equations from the AKNS hierarchy.

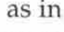
3. Breather, $\omega = \frac{4}{5}$ (Fig. 5).

$$\omega =$$

$$4 \tan^{-1}\left(\frac{\sqrt{1-\omega^2}}{\omega} \sin(\omega(x-y)) \operatorname{sech}(\sqrt{1-\omega^2}(x+y))\right)$$

$$0 < \omega < 1,$$

gives solutions that pulse in the $x - y$ direction and are localized in the $x+y$ direction — a breather. For irrational ω these surfaces intersect themselves infinitely often, but for rational ω's can close neatly, giving cusped surfaces with finite area.

4. Breather, $\omega = \frac{3}{5}$ (Fig. 6). Melko and Sterling [11] have made a detailed study of the relationship between the geometry of the sine–Gordon equation and that of surfaces of constant negative curvature. They point out that surfaces like the one here contain planar elastic curves, such as the Euler loop () — curves with stationary total square curvature, as in a bent plastic ruler. They also suggest an intriguing connection with the nonlinear Schrödinger equation: the surfaces swept out by curves obeying this equation [6] may form the boundary of the space of constant-negative-curvature surfaces.

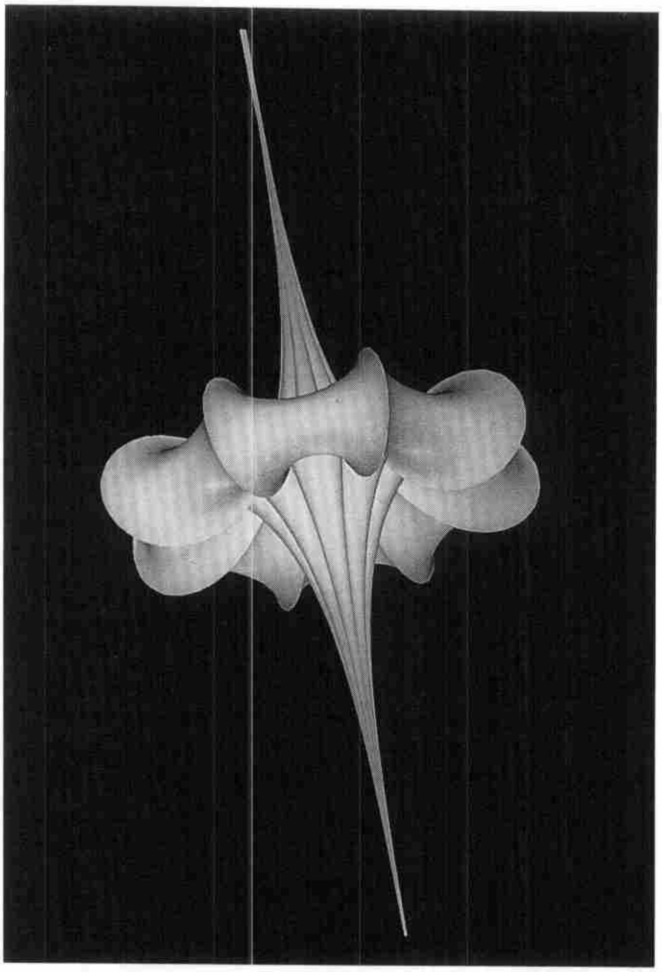


Figure 5.

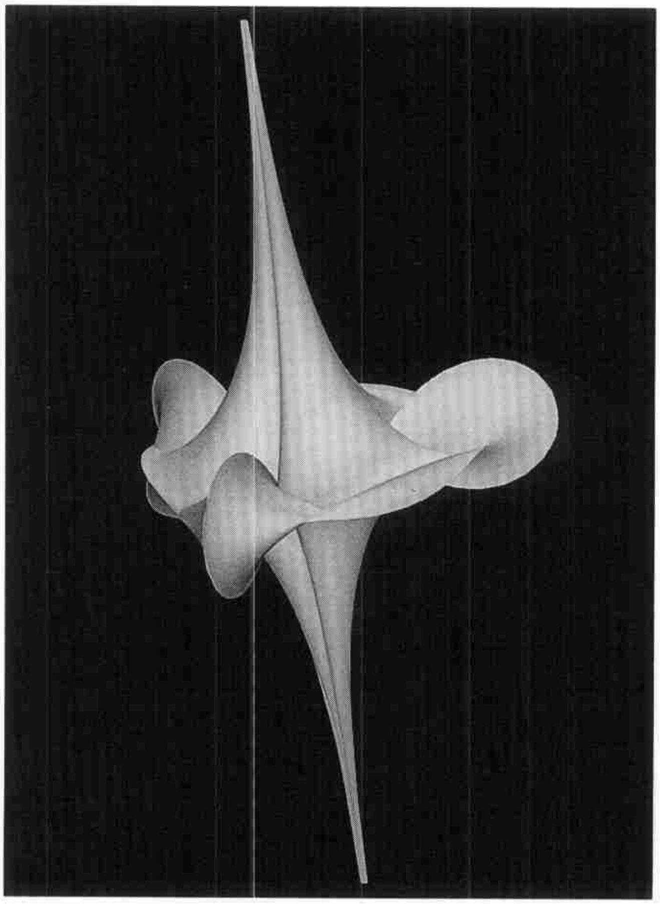


Figure 6.

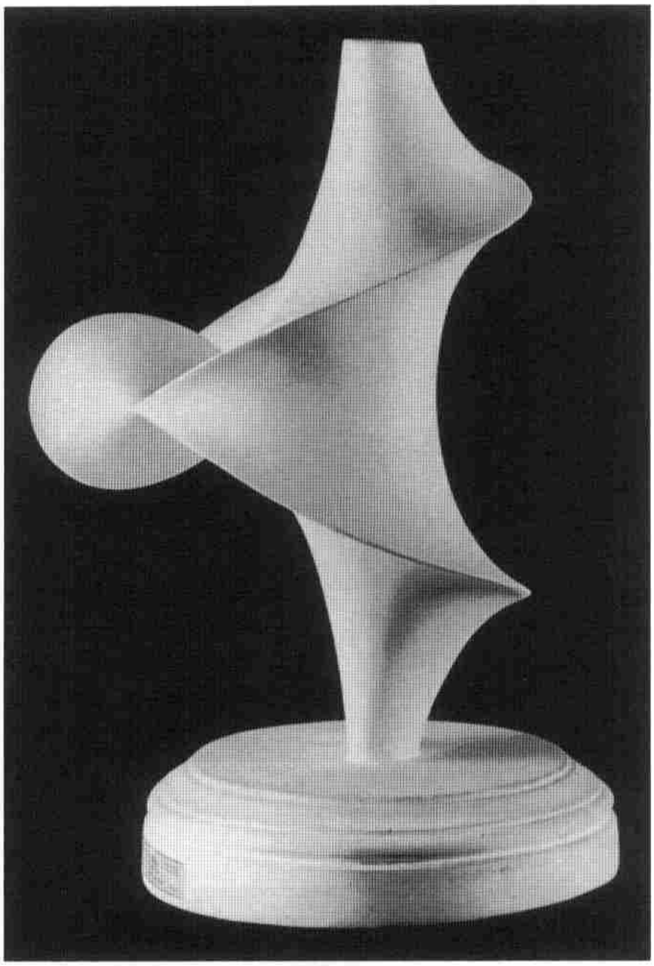


Figure 7. (from *Mathematische Modelle*, hrsg. Von Gerd Fischer. Vieweg, Braunschweig/Wiesbaden, 1986.)

5. Periodic Breather. There are similar breathers which are also periodic in $x + y$,

$$\omega = 4\ \tan^{-1}[a\ \mathrm{sn}(b(x-y)\,|\,k^2)\,\mathrm{dn}(c(x+y)\,|\,1-m^2)],$$

where $a^2 = k/m$ and $1/kb^2 = 1/mc^2 = (1+k^2)/k + (1+m^2)/m$. This contains many simpler solutions as special cases: k, $m \to 0$ with $m/k = \omega^2/(1-\omega^2)$ held constant recovers the regular breather with parameter ω; k, $m \to 1$ gives $\omega \to 4\ \tan^{-1}\ \tanh[(x-y)/2]$, which is another way of writing the pseudosphere; for $m \to 0$ with k fixed, $\omega \to 4\ \tan^{-1}[(x-y)\,\mathrm{sech}(x+y)]$, the kink–antikink solution [for which $\omega(x, y)$ is a clockwise helix as $x - y \to \infty$, zero at $x - y = 0$, and a counterclockwise helix as $x - y \to -\infty$].

The kink–antikink solution gives *Kuen's surface* (Fig. 7). This surface was found in terms of elementary functions in 1884 and a plaster model constructed, by what painstaking process one can only imagine, as part of the collection of mathematical models at the University of Göttingen. There's a photo of this collection in the Winter 1993 issue of *Mathematical Intelligencer*, p. 61, and the whole set is reviewed in *Mathematische Modelle* by Gerd Fischer [5]. Forget computer graphics—where else can you see a plaster model of the imaginary part of the Weierstrass-$\wp$ function?

In general, the periodic breather surfaces are rather complicated, but they are pretty in small pieces. In Figures 8–10 are three frames of a movie showing the pseudosphere awakening: $k = 0.9$ is fixed and $m = 0.43$, 0.11, and 0.01. It passes through a manta ray state and ends as a Calla lily. Edward Weston, where are you now?

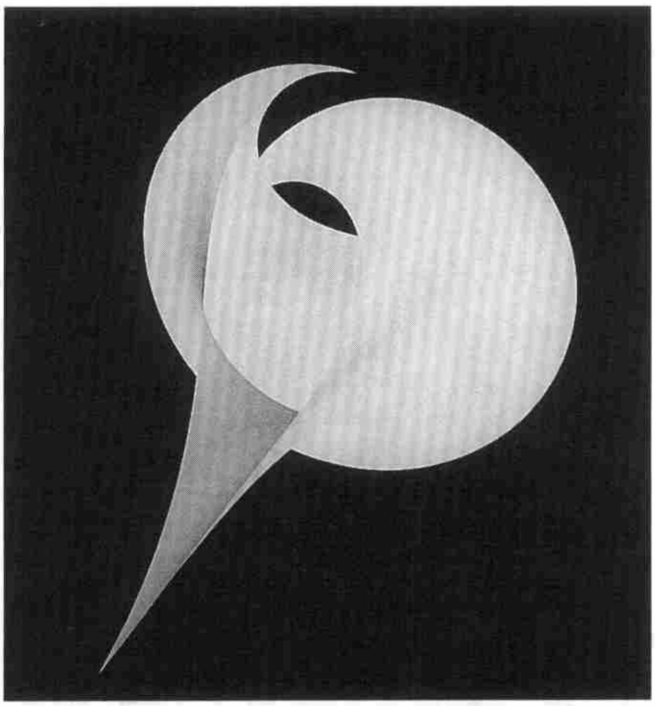


Figure 8.

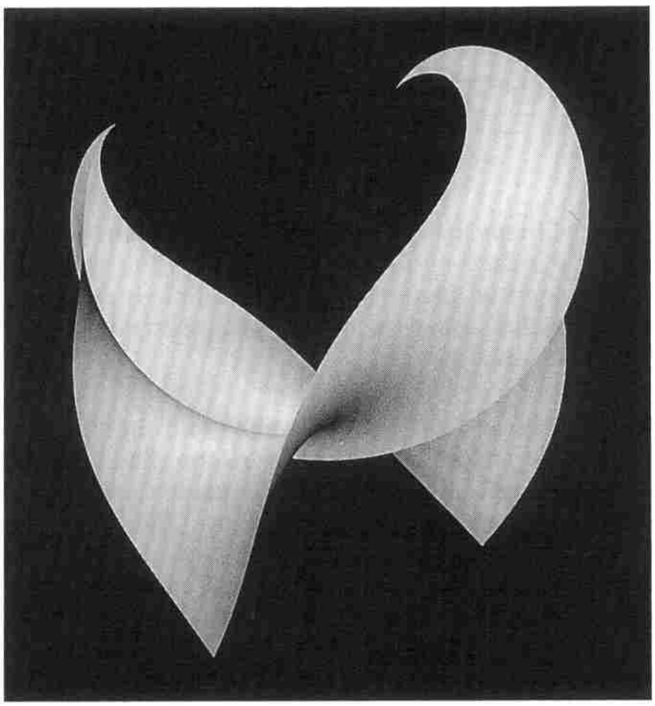


Figure 9.

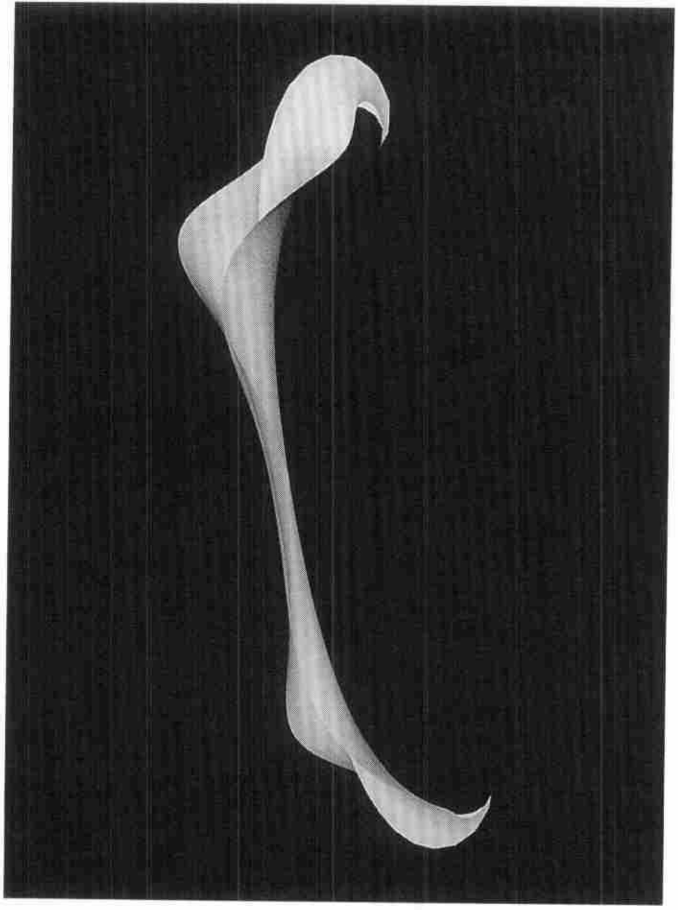


Figure 10.

Acknowledgments

I'd like to thank Jim Curry and Harvey Segur for their encouragement, and Stuart Levy at the Geometry Center, Minneapolis for his help with the final images.

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On the cut of clothes, by P. L. Tchebychef

(Association française pour l'avancement des sciences. 7th session. Paris. Meeting of 28 August 1878).

After having indicated that the idea of this study came from the communication made, two years ago, at the Congress of Clermont-Ferrand, by M. Edouard Lucas, on the geometry of weaving from fabric with rectilinear threads, M. Tchébichef set out general principles for determining the curves along which one can cut different pieces of material to make a well-adjusted sheath serving to envelop a body of any form.

Taking as a point of departure the principal observation that in the deformation of fabric one can consider, in the first approximation, the change of the angles formed by the warp threads and the weft threads, without taking account of the lengthening of the threads, he gave formulas which allow the determination of the outline of two, three, or four pieces of fabric to cover the surface of a sphere, with any desired accuracy. M. Tchébichef presented in this section a rubber ball covered by fabric, of which two pieces had been cut following his instructions; he observed that the problem would be essentially different if one replaced the fabric by a skin. Besides the proposed formulas, M. Tchébichef also gave a method to follow for the juxtaposition of pieces for sewing.

* * *

In accordance with the will of Tchebychef, the study "On the cut of clothes" found in his papers will not be published, because the manuscript did not bear the inscription *"publish."*

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