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Well-posedness of modified Camassa–Holm equations

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ABSTRACT

The Camassa–Holm equation can be viewed as the geodesic equation on some diffeomorphism group with respect to the invariant H^1 metric. We derive the geodesic equations on that group with respect to the invariant H^k metric, which we call the modified Camassa–Holm equation, and then study the well-posedness and dynamics of a modified Camassa–Holm equation on the unit circle S, which has some significant difference from that of Camassa–Holm equation, e.g., it does not admit finite time blowup solutions.

1. Introduction

In this paper, we study the Cauchy problem for modified Camassa–Holm equations derived as the Euler–Poincaré differential equation on the Bott–Virasoro group with respect to the H^k metric, i.e., the Cauchy problem for

$$m_t + um_x + 2u_x m = 0, \quad \text{with } m = \Lambda^{2k}u \equiv \left(I - \partial_x^2\right)^k u, \tag{1.1}$$

where $k \geqslant 2$ is a positive integer. This equation with $k = 0, 1$ corresponds to the KdV equation and the Camassa–Holm equation respectively.

In the study of shallow water waves, R. Camassa and D.D. Holm [4] derived in 1993 the following partial differential equation (PDE),

$$\left(I - \partial_x^2\right)u_t + 2\partial_x u \cdot \left(I - \partial_x^2\right)u + u \cdot \left(\partial_x - \partial_x^3\right)u = 0, \tag{1.2}$$

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and intensively studied its properties: its complete integrability, its bi-Hamiltonian structure, infinite conservation laws and the existence of peaked soliton solutions. For these reasons, this PDE is called Camassa–Holm equation and considered as one of the most fascinating PDEs in mathematical physics. Since the birth of this equation, many people have contributed to the well-posedness study on the whole real line $\mathbb{R}$ or on the unit circle S: to mention a few, Arnold and Khesin [2], Constantin and McKean [9], McKean [21], and the references therein. Local well-posedness for (1.2) was discussed by Constantin [6], Constantin and Escher [7] for the initial data in $H^s(S)$, $S = [0, 2\pi]$ with $s \geqslant 4$ and $s \geqslant 3$ respectively, and by Misiołek [22] with $s > 3/2$. Local well-posedness in the non-periodic case was proved for the initial data in $H^s(\mathbb{R})$ with $s > 3/2$ by Li and Olver [19] and Rodríguez-Blanco [23]. Classical solutions can become singular in finite time if the initial momentum $(I - \partial_x^2)u$ changes sign. The other related shallow water equations, such as the Benjamin–Ono equation, can be found in [13,14], etc. It is worthwhile to mention that Xin and Zhang [24] proved the global existence of the *weak solution* in the *energy space* $H^1(\mathbb{R})$ without any sign conditions on the initial value, and the uniqueness of this *weak solution* is obtained under some restrictions on the solution [25].

The ideas introduced by Arnold [1] which view the Euler fluid equation as the geodesic equation on some diffeomorphism group lead to a completely new stage of development of the Euler equation. Now the geodesic equations on Lie groups are called *generalised Euler equations*. Khesin and Misiołek [18] proved that the Camassa–Holm equation is the equation of the geodesic flow associated to $H^1(S)$ metric on the diffeomorphism group $\mathrm{Diff}(S)$ of the unit circle $S = [0, 2\pi]$, which is the Euler–Poincaré equation by using the Lagrangian associated with the H^1 metric for the fluid velocity, i.e., the Lagrangian as a function of the fluid velocity which is given by the quadratic form,

$$l(u) = \frac{1}{2} \int \left(u^2 + u_x^2\right) \mathrm{d}x.$$

The motivations of our study on (1.1) are as follows:

- Mathematically, KdV and Camassa–Holm have some significant differences in dynamics. For example, the Camassa–Holm equation leads to blowup in finite time for some initial values and admits smooth solutions for some other initials while for KdV [27] we have the global well-posedness for all smooth enough initial values. So it is natural to ask how the dynamics of the generalised Euler equations depends on the metric on the Lie algebra?
- D.D. Holm et al. [15] discussed the applications of the generalised Euler equations in the computational anatomy and mentioned that a smoother kernel than the inverse of $I - \triangle$ is used there. Mathematically, this means that we need to consider the dynamics of the generalised Euler equations of H^k metric other than H^1 as in the Camassa–Holm equation.

In this paper, we will first derive the modified Camassa–Holm equation with respect to the H^k metric using the abstract theory on the generalised Euler equation, then study the local and global well-posedness and weak solutions of the derived equations, some of which are quite different from those of Camassa–Holm equation. More specifically, we derive Eq. (1.1) on the unit circle S in Section 2. We study in Section 3 the well-posedness of the equation using Kato theory and then we prove that the derived equation for $k \geqslant 2$ admits no finite time blowup solution.

The local well-posedness can be stated as (all theorems are stated for $k = 2$):

Theorem 1.1. *Let $k = 2$, $u_0 \in H^s(S)$, $s > 7/2$. Then, there exist a $T > 0$ depending on $\|u_0\|_s$, and a unique solution*

$$u \in C\big([0, T], H^s(S)\big) \cap C^1\big([0, T], H^{s-1}(S)\big)$$

satisfying (1.1)*. Moreover, the map $u_0 \in H^s \mapsto u \in C([0, T], H^s(S))$ is continuous.*

The global well-posedness can be stated in terms of the momentum $m(= \Lambda^{2k}u)$:

Theorem 1.2. *Suppose $k \geqslant 2$ in (1.1). If the initial value $m(0,x) \in L^2(S)$, then there is a unique solution $m(t,x) \in L^2(S)$ for any finite time $t > 0$, and there exists a constant C_0 depending only on the norm of initial values u such that*

$$\|m\|_{L^2} \leqslant e^{C_0 t}\|m_0\|_{L^2}. \tag{1.3}$$

This is quite different from that of Camassa–Holm equation, because we know that for Camassa–Holm equation, even some very smooth initial values may lead to the finite time blowup solution, i.e., the momentum may blow up to ∞ in finite time.

Theorem 1.3. *Suppose $k = 2$, $u_0 \in H^s(S)$, $s > 7/2$, then Eq.* (1.1) *admits a unique solution in $C([0,+\infty), H^s(S)) \cap C^1([0,+\infty), H^{s-1}(S))$ if the initial momentum $m_0 \geqslant 0$.*

In Section 4, we enlarge the space of solutions to include the Dirac δ momentum and prove the existence and uniqueness of *weak solution* $u \in H^2(S)$ for an initial positive Radon measure momentum m_0:

Theorem 1.4. *Let $u_0 \in H^2(S)$, where $m_0 = (I - \partial_x^2)^2 u_0$ is a positive Radon measure on S. Then there exists a unique global* weak *solution $u \in C([0,\infty); H^2(S))$ of* (1.1) *and such that $m = \Lambda^4 u$ is a positive Radon measure on S whose total variation on S is uniformly bounded for $t \geqslant 0$. Moreover we have*

$$\int_S u\,\mathrm{d}x = \int_S u_0\,\mathrm{d}x, \qquad \int_S \left(u^2 + 2u_x^2 + u_{xx}^2\right)\mathrm{d}x = \int_S \left(u_0^2 + 2u_{0x}^2 + u_{0xx}^2\right)\mathrm{d}x. \tag{1.4}$$

Most of the notations in this paper are standard in the PDE field and can be found, e.g., in [12] and [28]. $\mathcal{B}(X,Y)$ denotes the space of all bounded linear operators from a Banach space X to a Banach space Y ($\mathcal{B}(X)$ if $X = Y$); $D = \partial = \partial_x = \frac{\partial}{\partial x}$; $\Lambda^s = (I - \partial_x^2)^{s/2}$, $s \in \mathbb{R}$; the standard Sobolev space $H^s = H^s(S)$ on the unit circle S with norm $\|\cdot\|_{H^s} = \|\cdot\|_s$ and $\langle,\rangle_s$ for its inner product; $H^{-s}(S) = (H^s(S))^*$ the dual space of $H^s(S)$, $H^\infty = \bigcap_{s\geqslant 0} H^s$; $[A,B] = AB - BA$ denotes the commutator of two linear operators A and B. $[\Lambda, f]g = \Lambda(fg) - f\Lambda g$ for functions f, g.

2. Derivation of the equations

We will use the following fundamental result about the geodesic flows on Lie groups to derive the PDE (1.1).

Proposition 2.1. *Let G be a (possibly infinite-dimensional) Lie group equipped with a metric $\langle\cdot,\cdot\rangle$ which is invariant under the right translations $R_g : G \mapsto G$, $R_g(h) = h \cdot g$. A curve $t \mapsto \gamma(t)$ in G is a geodesic of this metric if and only if $u(t) \equiv \mathrm{d}_{\gamma_t} R_{\gamma_t^{-1}} \dot\gamma(t)$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = -\mathrm{ad}^*_{u(t)} u(t), \tag{2.1}$$

where ad^*_u is the adjoint of ad_u with respect to the metric $\langle\cdot,\cdot\rangle$, that is for any u, v and $w \in T_eG$,

$$\left\langle \mathrm{ad}^*_u v, w\right\rangle_e = \left\langle v, [u,w]\right\rangle_e. \tag{2.2}$$

The proof of this proposition can be found in [2] or [20].

We need to introduce some notions on the Bott–Virasoro group before we derive the equations.

2.1. Bott–Virasoro group

Let $\mathcal{D}^s(S)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the unit circle S, and let $\mathrm{Vect}^s(S) = T_e\mathcal{D}^s(S)$ be the corresponding Lie algebra. We assume s to be large enough so that all our formal calculations can be rigorously justified.

The Bott–Virasoro group $\widehat{\mathcal{D}}^s(S)$ is the non-trivial central extension of $\mathcal{D}^s(S)$ which is defined as follows: the set

$$\widehat{\mathcal{D}}^s(S) \equiv \mathcal{D}^s(S) \times \mathbb{R}$$

and the group operation is defined by Bott [3]

$$\hat{\eta} \circ \hat{\xi} = \left(\eta \circ \xi, \alpha + \beta + \int_S \log \partial_x(\eta \circ \xi)\, \mathrm{d}\log \partial_x \xi\right), \tag{2.3}$$

where $\hat{\eta} = (\eta, \alpha)$, $\hat{\xi} = (\xi, \beta)$ with $\eta, \xi \in \mathcal{D}^s(S)$ and $\alpha, \beta \in \mathbb{R}$.

The corresponding Virasoro algebra $\widehat{\mathrm{Vect}}(S)$ is the tangent space of $\widehat{\mathcal{D}}^s(S)$ at the identity which is the non-trivial extension of $\mathrm{Vect}(S)$, the tangent space of $\mathcal{D}^s(S)$ at the identity of $\mathcal{D}^s(S)$. The commutator in the Lie algebra is given by [2]

$$[\widehat{V}, \widehat{W}] \equiv -\left((v\partial_x w - w\partial_x v)\frac{\partial}{\partial x}, c(v, w)\right), \tag{2.4}$$

where $c(v, w) \equiv \int_S v\partial_x^3 w\, \mathrm{d}x$, $\widehat{V} = (v\frac{\partial}{\partial x}, a)$, $\widehat{W} = (w\frac{\partial}{\partial x}, b)$ with $a, b \in \mathbb{R}$ and $v\frac{\partial}{\partial x}, w\frac{\partial}{\partial x} \in T_e\mathcal{D}^s(S)$.

2.2. Derivation of the equations

In order to derive Eq. (1.1), according to Proposition 2.1, the key point we need to find is the expression of ad^*.

Take $\widehat{U} = (u\frac{\partial}{\partial x}, a)$, $\widehat{V} = (v\frac{\partial}{\partial x}, b)$, $\widehat{W} = (w\frac{\partial}{\partial x}, c) \in \widehat{\mathrm{Vect}}^s(S)$, and define the H^k inner product on $\widehat{\mathrm{Vect}}^s(S)$ by

$$\langle \widehat{U}, \widehat{V}\rangle_{H^k} = \int_S \Lambda^k u \cdot \Lambda^k v\, \mathrm{d}x + ab, \tag{2.5}$$

where $\Lambda = (I - \partial_x^2)^{\frac{1}{2}}$, then we find $\mathrm{ad}^*_{\widehat{U}}$ by direct calculations

$$\begin{aligned}
\langle \mathrm{ad}^*_{\widehat{U}} \widehat{V}, \widehat{W}\rangle_{H^k} &= \langle \widehat{V}, \mathrm{ad}_{\widehat{U}} \widehat{W}\rangle_{H^k} = \langle \widehat{V}, [\widehat{U}, \widehat{W}]\rangle_{H^k} \\
&= \langle v, u_x w - u w_x\rangle_{H^k} - b \cdot c(u, w) \\
&= \left(\Lambda^{2k} v, u_x w - u w_x\right)_{L^2} - b \cdot c(u, w) \\
&= \left(g + b\partial_x^3 u, w\right)_{L^2} = \left\langle \Lambda^{-2k}\left(g + b\partial_x^3 u\right), w\right\rangle_{H^k},
\end{aligned} \tag{2.6}$$

where $g = 2u_x\Lambda^{2k} v + u\Lambda^{2k} v_x$. So

$$\mathrm{ad}^*_{\widehat{U}} \widehat{V} = \left(\Lambda^{-2k}\left(2u_x\Lambda^{2k} v + u\Lambda^{2k} v_x + b\partial_x^3 u\right)\frac{\partial}{\partial x}, 0\right). \tag{2.7}$$

The group $\widehat{\mathcal{D}}^s(S)$ is a right-invariant group, so by Proposition 2.1, the generalised Euler equation $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U} = -\mathrm{ad}^*_{\widehat{U}}\widehat{U}$ on the Virasoro group gives us

$$\frac{\mathrm{d}\Lambda^{2k}u}{\mathrm{d}t} = -\big(2u_x\Lambda^{2k}u + u\Lambda^{2k}u_x + a\partial_x^3 u\big), \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = 0, \tag{2.8}$$

which is (1.1) for $m = \Lambda^{2k}u$ if we take $a = 0$.

Remark. We can put Eq. (2.8) in the Hamiltonian form:

$$m_t = -\big(m\partial_x + \partial_x m + a\partial_x^3\big)\frac{\delta H}{\delta m}, \quad \text{with } H = \frac{1}{2}\int um\,\mathrm{d}x. \tag{2.9}$$

The KdV equation and the CH equation can be also put into this form (2.9) but with $m = u$ and $m = (1 - \partial_x^2)u$ respectively. On the other hand, we know that the KdV equation can be expressed as

$$m_t = -\partial_x\frac{\delta H_1}{\delta m}, \quad \text{with } m = u,\ H_1 = \frac{1}{2}\int\left(\frac{1}{3}u^3 - au_x^2\right)\mathrm{d}x, \tag{2.10}$$

and the Camassa–Holm equation

$$m_t = -\partial_x\big(1 - \partial_x^2\big)\frac{\delta H_1}{\delta m}, \quad \text{with } H_1 = \frac{1}{2}\int\big(u\big(u^2 + u_x^2\big) - au_x^2\big)\,\mathrm{d}x. \tag{2.11}$$

These equations give the second Hamiltonian structure for the KdV and CH equations respectively, where the term "bi-Hamiltonian structure" in some literature comes from, and the bi-Hamiltonian structure leads to the integrability of the equations and gives infinite conserved quantities for KdV and CH. Another interesting point is that they yield a constant Poisson structure $\mathcal{K} = \partial_x,\ \partial_x(1 - \partial_x^2)$ for KdV and CH, which is very much likely no longer true for the general $k > 1$ case.

3. Well-posedness

The a-term in (2.8) does not make much difference in the well-posedness study, so in this paper we focus on the limiting $a = 0$ case. The details in dealing with the general $a \neq 0$ case can be found in [26]. In this section, we will first use the Kato theory to establish the local well-posedness for (1.1), then we will prove the global well-posedness.

Theorem 3.1. *Let $k = 2$, $u_0 \in H^s(S)$, $s > 2k - \frac{1}{2} = 7/2$. Then, there exist a $T > 0$ depending on $\|u_0\|_s$, and a unique solution u satisfying* (1.1) *such that*

$$u \in C\big([0, T], H^s(S)\big) \cap C^1\big([0, T], H^{s-1}(S)\big).$$

Moreover, the map $u_0 \in H^s \mapsto u \in C([0, T], H^s(S))$ is continuous.

Remark. We state the theorem and give a proof for $k = 2$ only, but they are all valid for the general $k \geqslant 2$ case. The proof of this theorem consists of verifications of the conditions in Kato's Theorem [16] one by one, which we decompose as a series of lemmas.

We can rewrite (1.1) for $k = 2$ in two ways:

$$\begin{cases} m_t = -um_x - 2mu_x, & x \in S,\ t \in \mathbb{R}, \\ m(x, 0) = m_0(x) = \Lambda^4 u_0(x), \end{cases} \tag{3.1}$$

where $m = \Lambda^4 u = (I-\partial_x^2)^2 u,\ \Lambda^s = (I-\partial_x^2)^{\frac{s}{2}}$. Or

$$\begin{cases} u_t = -uu_x - \partial_x \Lambda^{-4}\big(u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x\partial_x^3 u\big), & x \in S,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{3.2}$$

If we denote $A(u) = u\partial_x,\ f \equiv -\partial_x\Lambda^{-4}(u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x\partial_x^3 u)$, then (3.2) has the form:

$$(C) \quad \begin{cases} \dfrac{\partial u}{\partial t} + A(u)u = f(u) \in X, & t \geqslant 0, \\ u(0) = u_0 \in Y. \end{cases} \tag{3.3}$$

We will verify that all conditions in Kato's Theorem [16] are satisfied, i.e., we need the following lemmas:

Lemma 3.2. *The operator $A(u) = u\partial_x$, with $u \in H^s$, $s > \frac{3}{2}$ belongs to $G(H^{s-1}, 1, \beta)$ for some $\beta > 0$.*

Lemma 3.3. *$B(u) = [\Lambda, u\partial_x]\Lambda^{-1} \in \mathcal{B}(H^{s-1})$ for $u \in H^s$, $s > 3/2$.*

Lemma 3.4. *For $u \in H^s(S)$ with $s > 3/2$,*

(a) $H^s \subset \mathcal{D}(u\partial_x) = \{f \in H^{s-1}\colon u\partial_x f \in H^{s-1}\}$, $s > 3/2$;
(b) $u\partial_x \in \mathcal{B}(H^s, H^{s-1})$, $s > 3/2$;
(c) $\|u\partial_x - v\partial_x\|_{\mathcal{B}(H^s,H^{s-1})} \leqslant C\|u-v\|_{s-1}$.

The proof of these three lemmas can be found, e.g., in [23].

In order to verify the condition (f_1) in [16] on the f-term, we need the following lemma on the estimate of product of two functions:

Lemma 3.5. *For any two functions f, g defined on S, we have*

(a) $\|fg\|_{H^t} \leqslant C\|f\|_{H^t}\|g\|_{H^t}$ *for* $t > \frac{1}{2}$;
(b) $\|fg\|_{H^t} \leqslant C\|f\|_{L^\infty}\|g\|_{H^t}$ *for* $t \leqslant 0$;
(c) $\|fg\|_{H^t} \leqslant C\|f\|_{H^{t+1/2}}\|g\|_{H^{t+1/2}}$ *for* $0 < t \leqslant \frac{1}{2}$;
(d) $\|fg\|_{H^t} \leqslant C(\|f\|_{L^\infty}\|g\|_{H^t} + \|g\|_{L^\infty}\|f\|_{H^t})$ *for* $t \geqslant 0$;
(e) $\|fg\|_{H^t} \leqslant C(\|f\|_{W^{t,\infty}}\|g\|_{L^2} + \|g\|_{H^t}\|f\|_{L^\infty})$ *for* $t \geqslant 0$.

Proof. (a) is the consequence of the fact that H^t is a Banach algebra for $t > \frac{1}{2}$.

(b) For $t \leqslant 0$, and any $h \in H^{-t}(S)$, we have

$$\begin{aligned} \left|\int_S fgh\,\mathrm{d}x\right| &\leqslant \|f\|_{L^\infty}\int |gh|\,\mathrm{d}x \\ &\leqslant \|f\|_{L^\infty}\|g\|_{H^t}\|h\|_{H^{-t}}, \end{aligned} \tag{3.4}$$

from which (b) follows.

(c) It is obvious.

(d) This estimate is from [17].

(e) Similar arguments as in [17] yield (e). □

Lemma 3.6. *Let $f(u) = -\partial_x\Lambda^{-4}(u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x\partial_x^3 u)$, $s > 7/2$, then*

(a) $\|f(u) - f(v)\|_{H^{s-1}} \leqslant C\|u-v\|_{H^{s-1}}$;
(b) $\|f(u) - f(v)\|_s \leqslant C\|u-v\|_s$.

Proof. (a) We need only to verify that

$$\left\|\partial_x\Lambda^{-4}\left(\frac{5}{2}u_{xx}^2+5u_x\partial_x^3u-\frac{5}{2}v_{xx}^2-5v_x\partial_x^3u\right)\right\|_{s-1}\leqslant C\|u-v\|_{s-1},$$

for the corresponding inequality for the rest terms is easier to verify.

$$\begin{aligned}\left\|\partial_x\Lambda^{-4}\left(u_{xx}^2-v_{xx}^2\right)\right\|_{s-1}&\leqslant C\left\|\partial_x^2(u+v)\partial_x^2(u-v)\right\|_{s-4}\\&\leqslant \max\left\{\left\|\partial_x^2(u+v)\right\|_{s-4},\left\|\partial_x^2(u+v)\right\|_{L^\infty},\left\|\partial_x^2(u+v)\right\|_{s-7/2}\right\}\\&\quad\cdot\max\left\{\left\|\partial_x^2(u-v)\right\|_{s-4},\left\|\partial_x^2(u-v)\right\|_{s-7/2}\right\}\quad\text{(by Lemma 3.5)}\\&\leqslant C\|u-v\|_{s-3/2}\leqslant C\|u-v\|_{s-1},\end{aligned}$$

$$\begin{aligned}\left\|\partial_x\Lambda^{-4}\left(u_x\partial_x^3u-v_x\partial_x^3v\right)\right\|_{s-1}&\leqslant C\left\|u_x\partial_x^3u-v_x\partial_x^3v\right\|_{s-4}\\&=C\left\|u_x\partial_x^3u-u_x\partial_x^3v+u_x\partial_x^3v-v_x\partial_x^3v\right\|_{s-4}\\&\leqslant C\left\|u_x\left(\partial_x^3u-\partial_x^3v\right)\right\|_{s-4}+\left\|(u_x-v_x)\partial_x^3v\right\|_{s-4}.\end{aligned}\tag{3.5}$$

We estimate these two terms separately. If $s-4>\frac{1}{2}$ or $-\frac{1}{2}<s-4\leqslant 0$, we can easily get from Lemma 3.5 that

$$\left\|u_x\left(\partial_x^3u-\partial_x^3v\right)\right\|_{s-4}\leqslant C\left\|\partial_x^3(u-v)\right\|_{H^{s-4}}\leqslant C\|u-v\|_{s-1}.\tag{3.6}$$

If $0<s-4\leqslant\frac{1}{2}$, we have to use Lemma 3.5 to get

$$\begin{aligned}\left\|u_x\partial_x^3(u-v)\right\|_{H^{s-4}}&\leqslant C\left(\|u_x\|_{L^\infty}\left\|\partial_x^3(u-v)\right\|_{H^{s-4}}+\|u_x\|_{W^{s-4,\infty}}\left\|\partial_x^3(u-v)\right\|_{L^2}\right)\\&\leqslant C\left(\|u\|_{W^{1,\infty}}\|u-v\|_{H^{s-1}}+\|u\|_{W^{s-3,\infty}}\|u-v\|_{H^3}\right)\\&\leqslant C\|u\|_{H^s}\|u-v\|_{H^{s-1}}\quad\text{because } s-4>0.\end{aligned}\tag{3.7}$$

Similarly, we can estimate the other term in (3.5). Here we just write out the formula for the case $0<s-4\leqslant\frac{1}{2}$,

$$\begin{aligned}\left\|(u_x-v_x)\partial_x^3v\right\|_{H^{s-4}}&\leqslant C\left(\|u_x-v_x\|_{s-4}\left\|\partial_x^3v\right\|_{L^\infty}+\left\|\partial_x^3v\right\|_{s-4}\|u_x-v_x\|_{L^\infty}\right)\\&\leqslant C\|v\|_{H^s}\|u-v\|_{H^{s-1}}.\end{aligned}\tag{3.8}$$

Adding up all the above estimates yields

$$\left\|f(u)-f(v)\right\|_{H^{s-1}}\leqslant C\|u-v\|_{H^{s-1}}.\tag{3.9}$$

(b) Similar argument as in (a).

$$\begin{aligned}\left\|\partial_x\Lambda^{-4}\left(u_{xx}^2-v_{xx}^2\right)\right\|_{s}&\leqslant C\left\|\partial_x^2(u+v)\partial_x^2(u-v)\right\|_{s-3}\\&\leqslant C\left\|\partial_x^2(u+v)\right\|_{s-3}\left\|\partial_x^2(u-v)\right\|_{s-3}\leqslant C\|u+v\|_{s-1}\|u-v\|_{s-1}\\&\leqslant C\|u-v\|_{s},\end{aligned}$$

$$\begin{aligned}\left\|\partial_x\Lambda^{-4}\left(u_x\partial_x^3u-v_x\partial_x^3v\right)\right\|_s &\leqslant C\left\|u_x\partial_x^3u-v_x\partial_x^3v\right\|_{s-3}\\ &= C\left\|u_x\partial_x^3u-u_x\partial_x^3v+u_x\partial_x^3v-v_x\partial_x^3v\right\|_{s-3}\\ &\leqslant C\|u\|_{s-2}\|u-v\|_s+C\|v\|_s\|u_x-v_x\|_{s-2}\\ &\leqslant C\|u-v\|_s,\end{aligned}$$

here we have used the fact that H^s is a Banach algebra for $s>1/2$. □

Proof of Theorem 3.1. Now Theorem 3.1 is just a direct consequence of Kato's Theorem with $Y=H^s(S)$, $X=H^{s-1}(S)$ and the above lemmas. □

Theorem 3.7. *If Theorem 3.1 yields the maximal time interval of existence is* $[0,T)$*, then we have* $T=+\infty$ *or*

$$\lim_{t\to T^-}\left\|u(t)\right\|_{H^s}=+\infty \quad \text{if } T<\infty. \tag{3.10}$$

Proof. From Theorem 3.1, we have $T=+\infty$ or

$$\lim_{t\to T^-}\left(\left\|u(t)\right\|_{H^s}+\left\|u_t(t)\right\|_{H^{s-1}}\right)=+\infty \quad \text{if } T<\infty. \tag{3.11}$$

On the other hand, we have from the proof of Theorem 3.1 and Eq. (3.2) that

$$\left\|u_t(t)\right\|_{H^{s-1}}\leqslant C\left\|u(t)\right\|_{H^s}, \tag{3.12}$$

which yields what we want. □

Based on the local well-posedness, some conservation laws can be established. In the following theorem, we assume that the solutions are smooth enough that all the calculations can be done rigorously.

Theorem 3.8. *Let* $u(x,t)$ *be the solution to* (1.1) *with* $u_0\in H^\infty$*, and* $m_0=(1-\partial_x^2)^2u_0$*, then in the time interval of existence of* u*, we have the following conserved quantities*:

$$I_1=\int m=\int u, \tag{3.13}$$

$$I_2=\int um=\int\left(u^2+2u_x^2+u_{xx}^2\right). \tag{3.14}$$

Proof. Integrating directly Eq. (1.1), we have the first conserved quantity. We can exploit Eqs. (3.1) and (3.2) to verify that

$$\frac{\mathrm{d}I_2}{\mathrm{d}t}=\int u_tm\,\mathrm{d}x+\int um_t\,\mathrm{d}x=0.$$

Geometrically, the fact that I_2 is conserved just means that velocity vector of the geodesic curve has a constant length along the geodesics. □

The Camassa–Holm equation (1.2) can reach a singularity in a finite time if $m_0=(I-\partial_x^2)u_0$ changes sign. However, this cannot happen for the modified equation (1.1) by our following theorem.

There are two unknowns u and m in (1.1) related by $m=\Lambda^{2k}u$, and here we choose to state the result in terms of m just for simplifying the statement.

Theorem 3.9. *Suppose $k \geqslant 2$ in (1.1). If the initial value $m(0,x) \in L^2(S)$, then $m(t,x) \in L^2(S)$ for any finite time $t > 0$, and there exists a constant C_0 depending only on the norm of initial value u_0 such that*

$$\|m\|_{L^2} \leqslant e^{C_0 t}\|m_0\|_{L^2}. \tag{3.15}$$

Proof. We prove Theorem 3.9 for sufficiently smooth function m and the general case $m_0 \in L^2$ follows by a standard density argument. Multiplying (1.1) by m and integrating over S, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m\|_{L^2}^2 + 2\int u_x m^2 + \int umm_x = a\int m\partial_x^3 u. \tag{3.16}$$

Clearly, $\int m\partial_x^3 u = \int \partial_x^3 u \Lambda^{2k} u = 0$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m\|_{L^2}^2 = -3\int m^2 u_x, \tag{3.17}$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m\|_{L^2}^2 \leqslant 3\|u_x\|_{L^\infty}\|m\|_{L^2}^2. \tag{3.18}$$

On the other hand, $I_2 = \int_S um\,\mathrm{d}x$ is a conserved quantity for (1.1), i.e.

$$\sum_{l=0}^{k} c_l \left\|\partial_x^l u(t,x)\right\|_{L^2}^2 = \sum_{l=0}^{k} c_l \left\|\partial_x^l u(0,x)\right\|_{L^2}^2, \tag{3.19}$$

with some positive constants depending on k and l. So from the Sobolev embedding theorem and $k \geqslant 2$ we have

$$\|u_x\|_{L^\infty} \leqslant C\|u_{xx}\|_{L^2} \leqslant C_0, \tag{3.20}$$

where C_0 is a constant depending only on the initial condition. The Gronwall inequality and (3.18) yield

$$\|m\|_{L^2} \leqslant e^{C_0 t}\|m_0\|_{L^2}. \qquad \square \tag{3.21}$$

Lemma 3.10. *Let $u(x,t)$ be the solution to (1.1) with $u_0 \in H^\infty$, and suppose that $m_0 = (1-\partial_x^2)^2 u_0 \geqslant 0$ (or $\leqslant 0$), then $m = (1-\partial_x^2)^2 u \geqslant 0$ (respectively $\leqslant 0$), moreover, if $m \geqslant 0$, then*

$$\int_S m^{1/2}\,\mathrm{d}x = \int_S m_0^{1/2}\,\mathrm{d}x.$$

Proof. The proof of Lemma 3.3 in [7] applies here with little change, and we omit it. $\square$

Remark. From the proof of Lemma 3.3 in [7], we can find that the essential part in the proof is the equation $m_t = -2um_x - u_x m$ and the conservation of $\int (u^2 + 2u_x^2 + u_{xx}^2)$ (which is $\int um$). The exact relation between m and u does not really matter as long as $\int |u_x|$ can be controlled by $\int um$.

Lemma 3.11. *Let $u_0 \in H^s(S)$, $s > 7/2$ and $m_0 = (1-\partial_x^2)^2 u_0 \geqslant 0$ (or $\leqslant 0$), then $\exists K > 0$ such that $\|u_{xxx}\|_{L^\infty} \leqslant K$.*

Proof. At first, we assume that $u_0 \in H^\infty$, u solves (1.1), then it is easy to show that $\|u\|_{L^2}^2 + 2\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2$ is conserved as long as u exists as a solution to (1.1). From Lemma 3.10, we have $m = \Lambda^4 u \geqslant 0$ (or $\leqslant 0$). Let $x_0 \in S$ satisfy $u_{xxx}(x_0) = 0$, then $\forall y \in S$, we have

$$\begin{aligned} u_{xxx}(y) &= \int_{x_0}^{y} \partial_x^4 u \, dx = \int_{x_0}^{y} \left(u - 2\partial_x^2 u + \partial_x^4 u\right) dx - \int_{x_0}^{y} \left(u - 2\partial_x^2 u\right) dx \\ &\leqslant \int_S m \, dx + \|u\|_{L^1} + 2\|u_{xx}\|_{L^1} = \int_S m_0 \, dx + \|u\|_{L^1} + 2\|u_{xx}\|_{L^1} \\ &\leqslant \int_S m_0 \, dx + \|u\|_{L^2} + 2C\|u_{xx}\|_{L^2} \leqslant K, \end{aligned} \tag{3.22}$$

where K depends on m_0 and $\|u_0\|_{H^2}$. Similarly, if we identify x_0 and $x_0 + 2\pi$ in S with $x_0 \leqslant y \leqslant x_0 + 2\pi$, then we have

$$-u_{xxx}(y) = \int_{y}^{x_0+2\pi} \partial_x^4 u \, dx \leqslant K.$$

So far we have proved the lemma for $u_0 \in H^\infty$. A standard approximation can give the proof for $u_0 \in H^s(S)$, $s > 7/2$. □

Theorem 3.12. *Suppose $k = 2$, $u_0 \in H^s(S)$, $s > 7/2$, then Eq.* (1.1) *with $a = 0$ admits a unique solution in $C([0, +\infty), H^s(S)) \cap C^1([0, +\infty), H^{s-1}(S))$ if the initial momentum $m_0 \geqslant 0$.*

This theorem holds also valid for $k \geqslant 2$ as long as $u_0 \in H^s(S)$ with $s > 2k - \frac{1}{2}$. In order to prove Theorem 3.12, we need the following lemma:

Lemma 3.13. *Assume the conditions in Theorem* 3.12 *hold true, then $\|u(t)\|_{H^s}$ is finite for any $0 < t < \infty$.*

Proof. Applying Λ^s to $u_t = -uu_x - f(u)$, where $f(u) = \partial_x \Lambda^{-4}(u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x\partial_x^3 u - a\partial_x^2 u)$, and multiplying by $\Lambda^s u$ and then integrating over S, we get

$$\frac{d}{dt}\|u\|_s^2 = -2\langle u, uu_x\rangle_s + \langle u, f(u)\rangle_s. \tag{3.23}$$

By the Kato–Ponce inequality [17], we have

$$\left|\langle u, uu_x\rangle_s\right| \leqslant C_s \|u_x\|_{L^\infty} \|u\|_s^2. \tag{3.24}$$

The Cauchy inequality gives

$$\left|\langle u, f(u)\rangle_s\right| \leqslant \|u\|_s \left\|f(u)\right\|_s, \tag{3.25}$$

and

$$\begin{aligned} \left\|f(u)\right\|_s &\leqslant C\left\|u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x\partial_x^3 u - a\partial_x^2 u\right\|_{H^{s-3}} \\ &\leqslant C\left(\left\|u^2\right\|_{s-3} + \left\|u_x^2\right\|_{s-3} + \left\|u_{xx}^2\right\|_{s-3} + \left\|u_x\partial_x^3 u\right\|_{s-3} + \|u\|_{H^s}\right) \end{aligned}$$

$$\begin{aligned}
&\leqslant C\big(\|u\|_{L^\infty}\|u\|_{s-3}+\|u_x\|_{L^\infty}\|u_x\|_{s-3}+\|u_{xx}\|_{L^\infty}\|u_{xx}\|_{s-3}\\
&\quad+\|u_x\|_{L^\infty}\big\|\partial_x^3u\big\|_{s-3}+\big\|\partial_x^3u\big\|_{L^\infty}\|u_x\|_{s-3}+\|u\|_{H^s}\big)\\
&\leqslant C\|u\|_s,
\end{aligned} \tag{3.26}$$

where we used again the Kato–Ponce inequality [17] and Lemma 3.11. So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_s^2 \leqslant C\|u\|_s^2, \tag{3.27}$$

and so the Gronwall's inequality completes the proof of the lemma. □

Proof of Theorem 3.12. Theorem 3.12 is a direct consequence of Theorem 3.7 and Lemma 3.13 above. □

4. Weak solutions

The previous well-posedness results assume $s > 2k - \frac{1}{2}$, which excludes the case $m = \delta$, the Dirac δ function. But we know that the δ function plays a very important role in the study of (generalised) Euler equations: both the point vortex in the Euler fluid equation and the peakon solution in the Camassa–Holm equation correspond to the δ momentum (or vortex). So in this section, we will study the *weak* solutions of the modified Camassa–Holm equation (1.1), among which is the δ momentum solution.

Eq. (3.2) can be rewritten as

$$\begin{cases} u_t + F(u)_x = 0, & x \in S,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

where

$$\begin{aligned}
F(u) &= \frac{1}{2}u^2 + \Lambda^{-4}\left(u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x\partial_x^3u\right)\\
&= \frac{1}{2}u^2 + \Lambda^{-4}\left(u^2 + 2u_x^2 + \frac{5}{2}u_{xx}^2\right) - 5\partial_x\Lambda^{-4}(u_xu_{xx}).
\end{aligned} \tag{4.2}$$

Definition 4.1. Let $u_0 \in H^2(S)$. A function $u : [0,+\infty) \times S \to \mathbb{R}$ is called a *global weak solution* to (4.1) if $u \in C([0,\infty); H^2)$ and $\forall T > 0$, we have

$$\int_0^T\int_S \big(u\varphi_t + F(u)\varphi_x\big)\,\mathrm{d}x\,\mathrm{d}t + \int_S u_0(x)\varphi(0,x)\,\mathrm{d}x = 0, \quad \forall \varphi \in C^{1,c}\big([0,T) \times S\big), \tag{4.3}$$

where $C^{1,c}([0,T) \times S)$ is the set of all first order smooth functions with compact support in $[0,T) \times S$.

Theorem 4.2. *Let $u_0 \in H^2(S)$, where $m_0 = (I - \partial_x^2)^2u_0$ is a positive Radon measure on S. Then there exists a unique global* weak *solution $u \in C([0,\infty); H^2(S))$ of (1.1) with $k = 2$, $a = 0$ such that $m = \Lambda^4u$ is a positive Radon measure on S whose total variation on S is uniformly bounded for $t \geqslant 0$. Moreover we have*

$$\int_S u\,\mathrm{d}x = \int_S u_0\,\mathrm{d}x, \qquad \int_S \big(u^2 + 2u_x^2 + u_{xx}^2\big)\,\mathrm{d}x = \int_S \big(u_0^2 + 2u_{0x}^2 + u_{0xx}^2\big)\,\mathrm{d}x. \tag{4.4}$$

Proof. Let $\theta \equiv \|m_0\|_{\mathcal{M}} = \|u_0 - 2\partial_x^2 u_0 + \partial_x^4 u_0\|_{\mathcal{M}}$ be the total variation of the Radon measure m_0, then by Lemma 5.2 in [8], there exist positive functions $m_0^n \in C^\infty(S)$ such that $\|m_0^n\|_{L^1} \leqslant C$ for a constant C depending on θ but independent of n, and $m_0^n \to m_0$ in $\mathcal{D}'(S)$. If we denote $u_0^n = \Lambda_4^{-4} m_0^n$, then $m_0^n = u_0^n - 2\partial_x^2 u_0^n + \partial_x^4 u_0^n$ and

$$\begin{aligned}\|u_0^n\|_{H^2}^2 = \int_S |u_0^n|^2 + 2|u_{0x}^n|^2 + |u_{0xx}^n|^2\,\mathrm{d}x \leqslant \left|\int_S m_0^n \cdot u_0^n\,\mathrm{d}x\right| \\ \leqslant \|m_0^n\|_{L^1}\|u_0^n\|_{L^\infty} \leqslant C\|m_0^n\|_{L^1}\|u_0^n\|_{H^1},\end{aligned} \tag{4.5}$$

which implies that

$$\|u_0^n\|_{H^2}^2 = \int_S |u_0^n|^2 + 2|u_{0x}^n|^2 + |u_{0xx}^n|^2\,\mathrm{d}x \leqslant C\|m_0^n\|_{L^1}^2 \leqslant C \tag{4.6}$$

for some constant dependent on θ. Then by Theorems 3.9 and 3.12 for the smooth initial value $u_0^n(x)$ there exists a unique solution $u^n \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1})$ to (4.1). We are going to use Arzelà–Ascoli Theorem to prove $\{u^n\}$ has a subsequence which is convergent in some sense. If we denote $m^n = u^n - 2\partial_x^2 u^n + \partial_x^4 u^n$, then

$$\|u^n(t)\|_{H^2} = \|u_0^n\|_{H^2} \leqslant C \quad \text{and} \quad \|m^n(t)\|_{L^1} = \|m_0^n(t)\|_{L^1} \leqslant C$$

because $m^n \geqslant 0$, where C is a constant independent of n. So

$$\|\partial_x^4 u^n\|_{L^1} \leqslant \|u^n\|_{L^1} + 2\|\partial_x^2 u^n\|_{L^1} + \|m^n(t)\|_{L^1} \leqslant C$$

and

$$\|\partial_x^3 u^n\|_{L^\infty} \leqslant C, \quad \text{with } C \text{ independent of } n.$$

So $\{u^n(t)\}$ is a compact set in $H^2(S)$ for any $t \geqslant 0$. On the other hand, $\|\frac{\mathrm{d}u^n}{\mathrm{d}t}\|_{H^2} = \|F(u^n)_x\|_{H^2}$ can be estimated as follows:

$$\begin{aligned}\|[(u^n)^2]_x\|_{H^2} = 2\|u^n u_x^n\|_{H^2} \leqslant C\big(\|u^n u_x^n\|_{L^2} + \|u_{xx}^n u_x^n\|_{L^2} + \|u^n u_{xxx}^n\|_{L^2}\big) \\ \leqslant C\|\partial_x^3 u^n\|_{L^2} \leqslant C\|\partial_x^3 u^n\|_{L^\infty} \leqslant C,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\left\|\partial_x \Lambda_4^{-4}\left(v^2 + 2v_x^2 - \frac{5}{2}v_{xx}^2 - 5v_x\partial_x^3 v\right)\right\|_{H^2} \leqslant C\left\|v^2 + 2v_x^2 - \frac{5}{2}v_{xx}^2 - 5v_x\partial_x^3 v\right\|_{H^{-1}} \\ \leqslant C \quad \text{if } v = u^n.\end{aligned} \tag{4.8}$$

So $\|\frac{\mathrm{d}u^n}{\mathrm{d}t}\|_{H^2} = \|F(u^n)_x\|_{H^2} \leqslant C$ with C independent of t and n. Therefore Arzelà–Ascoli Theorem tells us that $\{u^n(t)\}_{n\geqslant 1} \subset C([0,\infty); H^2)$ is a compact subset. So we have $u \in C([0,\infty); H^2)$ and $n_k \to \infty$ such that

$$u^{n_k} \to u \quad \text{in } C\big([0,\infty); H^2\big),$$

with

$$\left\|u(t)-u(s)\right\|_{H^2} \leqslant C|t-s|, \quad \forall t,s \geqslant 0,$$

$$u(0)=u_0.$$

Taking $n_k \to \infty$ in

$$\int_0^T \int_S \left(u^{n_k}\varphi_t + F\left(u^{n_k}\right)\varphi_x\right) \mathrm{d}x\,\mathrm{d}t + \int_S u_0^{n_k}(x)\varphi(0,x)\,\mathrm{d}x = 0, \quad \forall \varphi \in C^{1,c}\left([0,T)\times S\right), \tag{4.9}$$

yields that $u \in C([0,\infty); H^2)$ is the weak solution to (4.1).

From the proof above, we can easily get the conserved quantities and that the total variation $\|m(t,\cdot)\|_{\mathcal{M}}$ of the limit measure m satisfies

$$\left\|m(t,\cdot)\right\|_{\mathcal{M}} \leqslant \left\|m^n(t)\right\|_{L^1} = \left\|m_0^n(t)\right\|_{L^1} \leqslant C.$$

Uniqueness. Now we are proving the uniqueness of the solution. Here we just sketch the proof, and a rigorous argument can be realised by a standard mollification method. Let $G(x)$ be the Green's function for the operator $\Lambda^4 = (I-\partial_x^2)^2$ acting on $H^\infty(S)$, then from

$$\left(I - 2\partial_x^2 + \partial_x^4\right)G(x) = \delta(x) = \sum_{n=-\infty}^{\infty} e^{nix}, \tag{4.10}$$

we have

$$G(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+2n^2+n^4} e^{inx}$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+2n^2+n^4}\cos(nx), \quad x \in S. \tag{4.11}$$

Obviously, for any $0 \leqslant \varepsilon < 1$, $G(x) \in C^{2+\varepsilon}(S)$. Moreover, we have

Lemma 4.3.

$$\partial_x^3 G \in L^\infty(S).$$

Proof. We know from Abel's criterion that

$$\sum_{n=1}^{\infty} \frac{n^3}{1+2n^2+n^4}\sin(nx)$$

converges for any $x \in S$, and uniformly converges in any $[\alpha,\beta] \subset (0,2\pi)$ if $0 < \alpha < \beta < 2\pi$. That means the Fourier series

$$2\sum_{n=1}^{\infty} \frac{n^3}{1+2n^2+n^4}\sin(nx)$$

converges to $\partial_x^3 G$:

$$\partial_x^3 G(x) = 2\sum_{n=1}^{\infty} \frac{n^3}{1+2n^2+n^4}\sin(nx). \tag{4.12}$$

On the other hand, we know

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi}{2}\left(1-\frac{x}{\pi}\right) \quad \text{for } 0<x<2\pi, \tag{4.13}$$

and if we denote $g(x)$ for this function, then

$$\left|g(x)-\partial_x^3 G(x)\right| = \left|\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{n^3}{1+2n^2+n^4}\right)\sin(nx)\right| \tag{4.14}$$

which converges uniformly to a bounded function on S. So we have

$$\partial_x^3 G \in L^\infty(S). \qquad \square$$

Suppose now $u, v \in C([0,\infty); H^2)$ are two solutions of (3.2), i.e., they both solve the equation

$$\begin{cases} u_t = -uu_x - \partial_x \Lambda^{-4}\left(u^2+2u_x^2-\frac{5}{2}u_{xx}^2-5u_x\partial_x^3 u\right), & x\in S,\ t\in\mathbb{R}, \\ u(x,0)=u_0(x). \end{cases} \tag{4.15}$$

Or equivalently,

$$\begin{cases} u_t = -uu_x - G_x * \left(u^2+2u_x^2-\frac{5}{2}u_{xx}^2-5u_x\partial_x^3 u\right), & x\in S,\ t\in\mathbb{R}, \\ u(x,0)=u_0(x), \end{cases} \tag{4.16}$$

here $*$ stands for the convolution. Denote

$$M \equiv \sup_{t\geqslant 0}\left\{\left\|\Lambda^4 u\right\|_{\mathcal{M}} + \left\|\Lambda^4 v\right\|_{\mathcal{M}}\right\} < \infty, \tag{4.17}$$

then for all $(x,t)\in S\times\mathbb{R}_+$, we have

$$\begin{aligned} \left\|u(x,t)\right\|_{L^\infty} &= \|G*m\|_{L^\infty} \leqslant \|G\|_{L^\infty}\|m\|_{\mathcal{M}} \leqslant CM, \\ \left\|u_x(x,t)\right\|_{L^\infty} &= \|G_x*m\|_{L^\infty} \leqslant CM, \\ \left\|u_{xx}(x,t)\right\|_{L^\infty} &= \|G_{xx}*m\|_{L^\infty} \leqslant CM, \\ \left\|u_{xxx}(x,t)\right\|_{L^\infty} &= \|G_{xxx}*m\|_{L^\infty} \leqslant CM, \end{aligned} \tag{4.18}$$

and same estimates hold true for v as well.

Let $w=u-v$ and $A(u)=u^2+2u_x^2-\frac{5}{2}u_{xx}^2-5u_x\partial_x^3 u$, then

$$\begin{cases} w_t = -uw_x - wv_x - G_x*\left(A(u)-A(v)\right), & x\in S,\ t>0, \\ w|_{t=0}=0, & x\in S, \end{cases} \tag{4.19}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S |w|\,\mathrm{d}x = \int_S w_t \operatorname{sgn} w$$
$$= \int -u w_x \operatorname{sgn} w - w v_x \operatorname{sgn} w - G_x * \big(A(u) - A(v)\big) \operatorname{sgn} w, \tag{4.20}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S |w_x|\,\mathrm{d}x = \int_S w_{xt} \operatorname{sgn} w_x$$
$$= \int -\big[w_x(u_x + v_x) + u w_{xx} + w v_{xx}\big] \operatorname{sgn} w_x - G_{xx} * \big(A(u) - A(v)\big) \operatorname{sgn} w_x, \tag{4.21}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S |w_{xx}|\,\mathrm{d}x = \int_S w_{xxt} \operatorname{sgn} w_{xx}$$
$$= -\int \big[w_{xx}(2u_x + v_x) + w_x(2v_{xx} + u_{xx}) + u\partial_x^3 w + w\partial_x^3 v\big] \operatorname{sgn} w_{xx}$$
$$- \int G_{xxx} * \big(A(u) - A(v)\big) \operatorname{sgn} w_{xx}. \tag{4.22}$$

Using the estimates (4.18) for u, v, we have

$$\left|\int -u w_x \operatorname{sgn} w - w v_x \operatorname{sgn} w\right| \leqslant CM\left(\int |w| + |w_x|\right),$$
$$\left|\int \big[w_x(u_x + v_x) + u w_{xx} + w v_{xx}\big] \operatorname{sgn} w_x\right| \leqslant CM\left(\int |w| + |w_x| + |w_{xx}|\right),$$
$$\left|\int \big[w_{xx}(2u_x + v_x) + w_x(2v_{xx} + u_{xx})\big] \operatorname{sgn} w_{xx}\right| \leqslant CM\left(\int |w_x| + |w_{xx}|\right). \tag{4.23}$$

On the other hand, $A(u) - A(v) = w(u + v) + 2w_x(u_x + v_x) - \frac{5}{2} w_{xx}(u_{xx} + v_{xx}) - 5u_x\partial_x^3 w - 5w_x\partial_x^3 v - a\partial_x^2 w$, and integration by parts gives us

$$G_x * \big(u_x\partial_x^3 w\big) = G_{xx} * (u_x w_{xx}) - G_x * (u_{xx} w_{xx}),$$
$$G_{xx} * \big(u_x\partial_x^3 w\big) = G_{xxx} * (u_x w_{xx}) - G_{xx} * (u_{xx} w_{xx}), \tag{4.24}$$

which enables us to estimate

$$\left|\int G_x * \big(A(u) - A(v)\big) \operatorname{sgn} w\right| \leqslant CM\left(\int |w| + |w_x| + |w_{xx}|\right),$$
$$\left|\int G_{xx} * \big(A(u) - A(v)\big) \operatorname{sgn} w_x\right| \leqslant CM\left(\int |w| + |w_x| + |w_{xx}|\right). \tag{4.25}$$

The other terms in (4.22) can be estimated as follows:

$$\int u\partial_x^3 w \operatorname{sgn} w_{xx} = \int u \frac{\mathrm{d}}{\mathrm{d}x}|w_{xx}|\,\mathrm{d}x = -\int |w_{xx}| u_x\,\mathrm{d}x, \tag{4.26}$$

so

$$\left| \int u \partial_x^3 w \operatorname{sgn} w_{xx} \right| \leqslant M \int |w_{xx}|. \tag{4.27}$$

It is easy to see

$$\left| \int w \partial_x^3 v \operatorname{sgn} w_{xx} \right| \leqslant \left\| \partial_x^3 v \right\|_{L^\infty} \int |w| \leqslant M \int |w|. \tag{4.28}$$

In order to estimate $\int G_{xxx} * (A(u) - A(v)) \operatorname{sgn} w_{xx}$, we need only estimate $\int G_{xxx} * (u_x \partial_x^3 w)$ because the other terms can be estimated in the same way as the above terms. Again, the integration by parts yields

$$\begin{aligned} G_{xxx} * \left(u_x \partial_x^3 w\right) &= G_{xxxx} * (u_x w_{xx}) - G_{xxx} * (u_{xx} w_{xx}) \\ &= G_{xx} * (u_x w_{xx}) - G * (u_x w_{xx}) + u_x w_{xx} - G_{xxx} * (u_{xx} w_{xx}), \end{aligned} \tag{4.29}$$

here, we have used the definition of G, which gives us

$$G_{xxxx} * f - 2G_{xx} * f + G * f = f.$$

Now it is clear that

$$\left| \int G_{xxx} * \left(u_x \partial_x^3 w\right) \right| \leqslant CM \int |w_{xx}|. \tag{4.30}$$

Taking all the above estimates in account, we have

$$\frac{d}{dt} \int_S \left(|w| + |w_x| + |w_{xx}|\right) dx \leqslant CM \int_S \left(|w| + |w_x| + |w_{xx}|\right) dx, \tag{4.31}$$

and so the Gronwall's inequality yields $w \equiv 0$. This completes the proof of Theorem 4.2. □

5. Some remarks

5.1. The whole real line case

We have discussed the well-posedness of Eq. (1.1) on the periodic case. Actually, some of the above results hold true with $\Lambda^{2k} = (1 - \partial_x^2)^k$ on the whole real line case:

$$m_t + 2u_x m + u m_x = 0 \quad \text{on } \mathbb{R}, \text{ with } m = \left(1 - \partial_x^2\right)^k u. \tag{5.1}$$

More specifically, the local well-posedness Theorem 3.1 holds true if $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$, combining our arguments here and those estimates established for $(1 - \partial_x^2)$ in [23]. Theorem 3.9 with $m_0 \in L^2(\mathbb{R})$ and $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ holds true. Using Lemma 5.1 we will prove on the next page, we can prove that Theorem 3.12 holds true for (5.1) if we suppose $m_0 \geqslant 0$ and $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with some $s > 2k - \frac{1}{2}$. Theorem 4.2 holds true for (5.1) if $u_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $m_0 = (1 - \partial_x^2)^2 u_0$ a positive Radon measure.

In the case of $\mathbb{R}$, we can prove

(a) $G(x) \geqslant 0$ ($x \in \mathbb{R}$) for the fundamental solution $G(x)$ of the operator $(1 - \partial_x^2)^k$ on $\mathbb{R}$;

(b) $|\partial_x^{2k-1} G|_{L^\infty} < \infty$;
(c) An analogous lemma to Lemma 3.11.

In fact, the Green's function of Λ^{2k} on $\mathbb{R}$, $k \geqslant 1$, is given by

$$G(x) = \frac{1}{2k}\left(1 + |x| + |x|^2 + \cdots + |x|^{k-1}\right)e^{-|x|}, \tag{5.2}$$

so the items (a) and (b) are obvious. We just prove a lemma analogous to Lemma 3.11 (take $k = 2$ as an example).

Lemma 5.1. *Let $u_0 \in H^s(\mathbb{R})$, $s > 7/2$, $m_0 = (1 - \partial_x^2)^2 u_0 \geqslant 0$ (or $\leqslant 0$) smooth enough and $u_0 \in L^1(\mathbb{R})$, then $\exists K > 0$ such that $\|u_{xxx}\|_{L^\infty} \leqslant K$.*

Proof. From the assumption $m_0 = (1 - \partial_x^2)^2 u_0 \geqslant 0$, we can prove that $m(x,t) \geqslant 0$ for any $t \geqslant 0$ using a similar argument to that of Lemma 3.10, so we have $u = G * m \geqslant 0$ because $G(x) > 0$. From (5.1), we have

$$\left\|u(t,\cdot)\right\|_{L^1(\mathbb{R})} = \left\|m(t,\cdot)\right\|_{L^1(\mathbb{R})} = \left\|m_0(t,\cdot)\right\|_{L^1(\mathbb{R})}, \tag{5.3}$$

and the conservation law

$$\int um = \int_{\mathbb{R}} \left(u^2 + 2u_x^2 + u_{xx}^2\right) \mathrm{d}x = \int_{\mathbb{R}} \left(u_0^2 + 2u_{0x}^2 + u_{0xx}^2\right) \mathrm{d}x, \tag{5.4}$$

which implies

$$\|u_x\|_{L^\infty} \leqslant C \tag{5.5}$$

by the Sobolev embedding theorem.

On the other hand, because $m \geqslant 0$, $u \geqslant 0$, we have

$$0 \leqslant \int_{-\infty}^{x} m \,\mathrm{d}x = \int_{-\infty}^{x} \left(u - 2\partial_x^2 u + \partial_x^4 u\right) \mathrm{d}x \leqslant \|u\|_{L^1} - 2u_x + \partial_x^3 u, \tag{5.6}$$

$$\|u\|_{L^1} = \|m\|_{L^1} \geqslant \int_{-\infty}^{x} \left(u - 2\partial_x^2 u + \partial_x^4 u\right) \mathrm{d}x \geqslant -2u_x + \partial_x^3 u, \tag{5.7}$$

which implies

$$\left\|2u_x - \partial_x^3 u\right\|_{L^\infty} \leqslant \|u\|_{L^1}. \tag{5.8}$$

So combining Eqs. (5.5) and (5.8), we have

$$\left\|\partial_x^3 u\right\|_{L^\infty} \leqslant C \tag{5.9}$$

with a constant C depending only on the L^1 norm and H^2 norm of the initial u_0. □

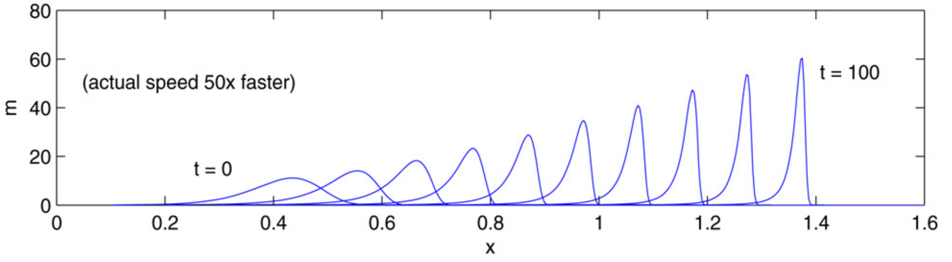

Fig. 1. The evolution of Gaussian initial value.

5.2. Numerical simulations and solitons

Our results in this paper tell us that the modified Camassa–Holm equation with $k \geqslant 2$ does not have finite time blowup solutions. Our numerical simulation, however, strongly suggests that some initial values evolve into a δ momentum as $t \to \infty$, which we call *weak blowup* to form a *soliton*. See Fig. 1 for the evolution of the Gaussian initial value m_0. One can see that maximum value of the solution m increases almost *linearly* in t.

This is very interesting because (a) the soliton is formed at $t = \infty$ and (b) we do not know if the PDE (1.1) is completely integrable, actually we tend to believe the PDE is not integrable with some strong support from our numerical experiments: we have studied in [26] the Lyapunov exponents of a four particle systems corresponding to (1.1), and the numerical result shows that there is at most *one* positive Lyapunov exponent, which indicates that the PDE is not integrable and there is much likely another conserved quantity in addition to $\int u$ and $\int um$ although we have not yet found any proper candidate for that at the moment. The stability of peakons for Camassa–Holm equation was demonstrated in [5] and studied in [10,11] with the help of the first three conserved quantities of Camassa–Holm equation. It seems that the solitons for generalised Camassa–Holm equations are also stable, although we have found only two conserved quantities. We will analyse the formation of the soliton out from (1.1) in a forthcoming paper.

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