Trends in Mathematics, © 1999 Birkhäuser Verlag Basel/Switzerland

On a possible mechanism of anomalous diffusion in geophysical turbulence

F. Dupont, R. I. McLachlan, V. Zeitlin

The main characteristic features of the geophysical turbulence are differential (so called β-effect) rotation and stratification. Each of these phenomena is responsible for a specific type of waves: planetary or Rossby waves and internal gravity waves, respectively. It is well-known that e.g. for the β- plane turbulence [1] one may have vortex-dominated (i.e. close to the 2d turbulence), wave-dominated or crossover regimes depending on the value of the characteristic nonlinearity parameter. Hence, if the Lagrangian transport in geophysical turbulence is studied the waves will play a rôle and it is important to know what are, precisely, their transport properties. In this connection it is known that, e.g., electromagnetic waves of sufficiently large amplitude in plasma may trap and effectively transport charged particles [2]. The crucial difference with fluid dynamics, however, is that nonlinear interaction of electromagnetic waves is weak even for large wave amplitudes and, usually, may be safely neglected while it is not the case for the above-mentioned waves. Thus, it is necessary to include nonlinear effects which will be done perturbatively in what follows. Note that previously the nonlinear interactions among waves were not taken into account neither in the studies of mixing by finite number of waves [3, 4, 5] nor in the numerical studies of diffusion by an ensemble of waves [6, 7].

Below we shall limit ourselves to Rossby waves, the conclusions for internal gravity waves being similar. The dynamical system we are interested in is an equation for the streamfunction ψ of the two-dimensional velocity field $\mathbf{v} = (-\partial\psi/\partial y, \partial\psi/\partial x)$

$$\frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi) + \beta\frac{\partial\psi}{\partial x} = 0 \tag{1}$$

where β is the Coriolis parameter, $J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\mathbf{x} = (x, y)$ are Cartesian coordinates on the β-plane. The linear part of this equation describes propagation of Rossby waves of the form $\psi = a\cos(\omega t - \mathbf{k}\cdot\mathbf{x} + \varphi)$ with dispersion relation $\omega = \Omega(k_x, k_y) = -\beta\frac{k_x}{\mathbf{k}^2}$ As is well known [8], in the lowest order of the perturbation expansion in wave amplitude the Rossby waves' interactions are dominated by resonant triads. Indeed, if one takes a triad of waves

$$\psi_1 = \sum_{i=1}^{3} a_i \cos\Phi_i \tag{2}$$

satisfying the resonance conditions

$$\Phi_1 + \Phi_2 = \Phi_3 \quad \forall \mathbf{x},\ t, \tag{3}$$

where

$$\Phi_i = \Omega_i t - \mathbf{k}_i \cdot \mathbf{x}, \tag{4}$$

$\Omega_i = \Omega(k_{x_i}, k_{y_i})$, ϵ is the small parameter, and $a_i = a_i(\tau)$ where $\tau = \epsilon^{-1} t$ is the slow modulation time, as a first-order approximation to the perturbative expansion $\psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots$ of the solution of (1), then the integrability conditions for the next approximation, ψ_2, give modulation equations for the wave amplitudes which admit solutions in terms of elliptic functions

$$a_1(\tau) = a_{1_0}\,\mathrm{dn}(\tau|\kappa), \quad a_2(\tau) = a_{2_0}\,\mathrm{sn}(\tau|\kappa), \quad a_3(\tau) = a_{3_0}\,\mathrm{cn}(\tau|\kappa). \tag{5}$$

In terms of the initial amplitudes and interaction parameters the modulus κ of the elliptic functions is given by

$$\kappa^2 = \frac{a_{3_0}^2}{a_{1_0}^2}\left|\frac{\gamma_1}{\gamma_2}\right|, \tag{6}$$

where γ_i, $i = 1, 2, 3$ are the interaction parameters

$$\gamma_1 = \frac{\mathbf{k}_2 \times \mathbf{k}_3}{\mathbf{k}_1^2}, \quad \gamma_2 = \frac{\mathbf{k}_3 \times \mathbf{k}_1}{\mathbf{k}_2^2}, \quad \gamma_3 = \frac{\mathbf{k}_1 \times \mathbf{k}_2}{\mathbf{k}_3^2}, \tag{7}$$

and the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a} \times \mathbf{b} = a_x b_y - a_y b_x$. The modulation period, T_M, is equal to the period $K(\kappa)$ of the elliptic functions, where K is the complete elliptic integral of the first kind. Note that one may solve explicitely the resonance conditions (3) and parameterize the triad space by two angles in the wavevectors' triangle, namely by α, the angle formed by $\mathbf{k}_1$ and $\mathbf{k}_3$, and by γ, that formed by $\mathbf{k}_2$ and $\mathbf{k}_3$ – a parameterization which will be used to explore the triads' space in what follows.

Advection of the passive tracer by the divergenceless velocity field associated with a streamfunction ψ is given by

$$\dot{x}(t) = -\frac{\partial\psi(x,y;t)}{\partial y};\ \dot{y}(t) = \frac{\partial\psi(x,y;t)}{\partial x}. \tag{8}$$

where $x(t), y(t)$ denote the particle position at time t. These equations represent a "one and a half" degree of freedom Hamiltonian system which may be studied by standard methods once ψ, the Hamiltonian, is known as a function of x, y and t. Before considering advection by resonant triads (2) we first summarize the dynamics of (8) in the presence of two harmonic waves. In this case

$$\psi = A_1 \cos(\Omega_1 t - \mathbf{k}_1{\cdot}\mathbf{x}) + A_2 \cos(\Omega_2 t - \mathbf{k}_2{\cdot}\mathbf{x}) \tag{9}$$

equations (8) are integrable as the time dependence may be eliminated by choosing the phases $\Phi_{1,2}$ as new dynamical variables (provided the matrix $K = \begin{pmatrix} k_{1_x} & k_{1_y} \\ k_{2_x} & k_{2_y} \end{pmatrix}$ is nonsingular). These variables are defined on the torus $\mathbf{T}^2$ but may be lifted to its universal cover $\mathbf{R}^2$. We get a system of equations

$$\begin{aligned} \dot{\Phi}_1 &= \Omega_1 + (\mathbf{k}_1 \times \mathbf{k}_2) A_2 \sin \Phi_2 \\ \dot{\Phi}_2 &= \Omega_2 - (\mathbf{k}_1 \times \mathbf{k}_2) A_1 \sin \Phi_1 \end{aligned} \tag{10}$$

which is an integrable canonical one-degree-of-freedom Hamiltonian system with Hamiltonian

$$H = \Omega_1 \Phi_2 - \Omega_2 \Phi_1 - (\mathbf{k}_1 \times \mathbf{k}_2)(A_1 \cos \Phi_1 + A_2 \cos \Phi_2). \tag{11}$$

We call $\mathbf{x}$ *physical space*, and $\Phi = (\Phi_1, \Phi_2)$ *phase space.* They are related by

$$\mathbf{x} = (K^{-1}\Omega)t - \Phi. \tag{12}$$

If a particle has $\mathbf{x}(t)$ bounded we call it *frozen*; if it is not bounded, but $\lim_{t\to\infty} \mathbf{x}(t)/t = 0$, we say it is *diffusive* (and, generally, $\mathbf{x}(t) \sim t^{1/2}$); if a non-zero $\lim_{t\to\infty} \mathbf{x}(t)/t$ exists , we say the motion is *ballistic.* Ballistic motion corresponds to particles being advected with, on average, a constant velocity; we call these particles *shooters.* In general, the situation where $\mathbf{x}(t) \sim t^{\alpha}$ with $\alpha \neq 1/2$ is known as anomalous diffusion. For advection by a pair of harmonic waves (9 - 11) the motion in phase space is along the isoenergetic curves $H = const$ with an average (over $\Phi \in \mathbf{T}^2$) velocity $\Omega = (\Omega_1, \Omega_2)$ which gives no average (over Φ) displacement in physical space. However, two different situations may take place according to the topology of the isoenergetic curves which is determined by the interplay of the linear and nonlinear terms in (11). In the case when the linear term is dominating the isolines of H are open and each point in the phase space is moving with a velocity close to the average velocity. In physical space any particle therefore remains close to its initial position and is frozen. In the case when linear and nonlinear terms are comparable, fixed points, each with a surrounding island of periodic orbits (elliptic islands) appear in the phase space. A fixed point in Φ-space, by virtue of (12), corresponds to a ballistically advected particle in physical space. It is clear that the whole elliptic island around a given fixed point will be also ballistically advected. The relative number of shooters is, thus, defined by the fraction of phase space occupied by the elliptic islands. All other points moving along the open orbits in phase space stay close to the origin in the physical space; but, in order to maintain zero average velocity, the cloud of "normal" points must drift in the direction opposite to the shooters' motion.

If confirmed, the ballistic advection phenomenon would provide a mechanism for the anomalous diffusion in geophysical turbulence. However, the condition for the appearance of shooters means that the wave amplitudes are relatively large and, thus, it is necessary to take into account the nonlinear effects. We therefore consider a resonant triad engendered by an initial pair of waves. The resonance condition (3) allows us to eliminate the third phase Φ_3 and, again, to get a Hamiltonian system defined in the phase space (Φ_1, Φ_2). The equations of motion and the Hamiltonian are, respectively:

$$\begin{aligned} \dot{\Phi}_1 &= \Omega_1 + (\mathbf{k}_1 \times \mathbf{k}_2)\,(a_2(t)\sin\Phi_2 + a_3(t)\sin(\Phi_1 + \Phi_2)) \\ \dot{\Phi}_2 &= \Omega_2 - (\mathbf{k}_1 \times \mathbf{k}_2)\,(a_1(t)\sin\Phi_1 - a_3(t)\sin(\Phi_1 + \Phi_2)) \end{aligned} \qquad (13)$$

$$H = \Omega_1\Phi_2 - \Omega_2\Phi_1 - (\mathbf{k}_1 \times \mathbf{k}_2)\,(a_1(t)\cos\Phi_1 + a_2(t)\cos\Phi_2 + a_3(t)\cos(\Phi_1 + \Phi_2)) \qquad (14)$$

and represent a one-and-a-half degree of freedom dynamical system which is not integrable due to the explicit time-dependence of the a_i. Note that the modulation $a_i(t)$, although periodic with period T_M, is not harmonic, except for the degenerate case of the isosceles triads . A standard method of investigation for this type of dynamical system is a numerical integration and a study of the phase portrait resulting from the iterations of the Poincaré map f_P over the modulation period T_M.

In general, for the non-integrable system (13) one might expect a chaotic sea with, possibly, some islands of regular behavior as a typical phase portrait. As $\beta \to \infty$ is an integrable limit of (13), one might also expect that for any triad a more and more regular pattern of dynamical behavior emerges with increasing β. As in the integrable case of a pair of waves, the presence of elliptic islands would signify a presence of shooters. Indeed, suppose the Poincaré map $f_P : \mathbf{T}^2 \to \mathbf{T}^2$ of (13) has a period n fixed point Φ_0 with a winding number $\mathbf{m}$ about the torus and n is the discrete time. This means that in terms of the lift map $\tilde{f}_P$ on $\mathbf{R}^2$ we have

$$\tilde{f}_P^n(\Phi_0) = \Phi_0 + 2\pi\mathbf{m} \qquad (15)$$

and the distance travelled in time nT_M in physical space is

$$\mathbf{x} = K^{-1} \cdot (\Omega n T_M - 2\pi\mathbf{m}) + \mathbf{x}(0). \qquad (16)$$

Hence, the fixed points and the regular orbits around them belonging to the stability islands having the same average winding number correspond to shooters. Chaotic orbits correspond to slow diffusion. The presence of stability islands results in a slow drift in physical space of the diffusive cloud away from the shooters, so as to compensate for the fast escape of the shooters and to keep the average winding number zero.

We performed a direct numerical investigation of (13) for a large range of triads using the above-mentioned explicit parametrization of these latter. In order to be consistent with perturbation theory we were trying to minimize the ratio

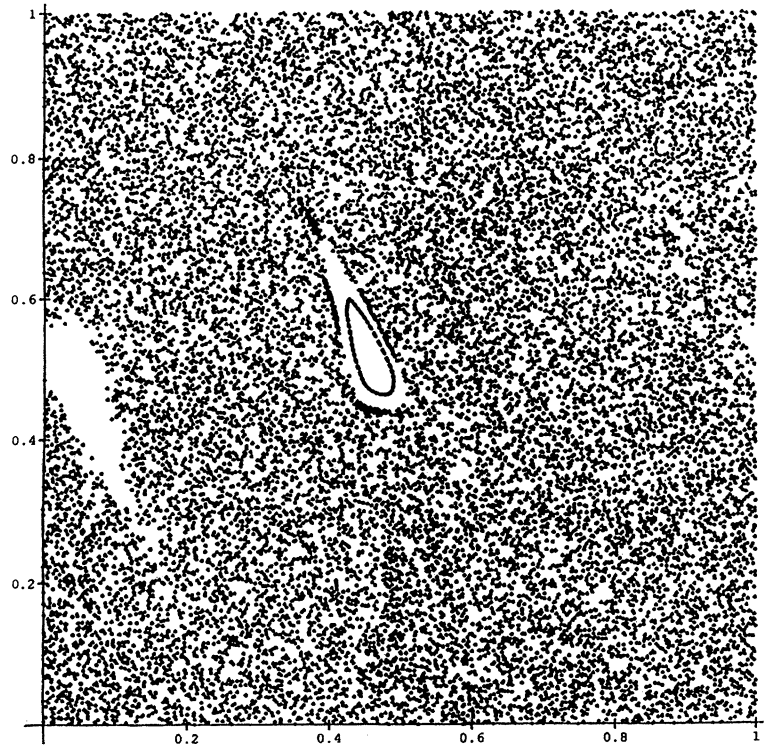

Figure 1: The phase portrait of the triad $\gamma = 0.7, \alpha = 1$ at $\beta = .5$ after 2000 T_M; $T_M = 47.14$. The wave periods T_i are 22.66, 10.98, 21.27, respectively. An elliptic trajectory inside the island is also shown.

$\frac{\max(T_i)}{T_M}$ by working in the correspondingly chosen region of the triads' parameters space. The smallness of this ratio may be considered as a rough criterion of applicability of the resonant triad description of the wave field generated by an initially active pair of waves 1 and 3. As we wish to follow many orbits of (13) for long times at many different parameter values we use a symplectic integrator [9], thus ensuring that the computed Poincaré map f_P is exactly (up to roundoff error) area-preserving and that no spurious non-Hamiltonian-like dynamics or bifurcations will be induced. We, generically, observe two types of scenario. The first one consists in a direct transition to quasi-uniform chaos from the regular behavior while decreasing β. The second one is characterized by a non-mixing ergodic behavior for large β, as expected, which gives place to chaos by more and more vigorous distortion of the phase trajectories and, then, to the birth of multiple relatively small and/or narrow islands. These primary islands disappear giving rise to chaos from which secondary "fat" islands are born (Fig. 1). It is these fat islands that give rise to relatively large numbers of shooters and a corresponding large

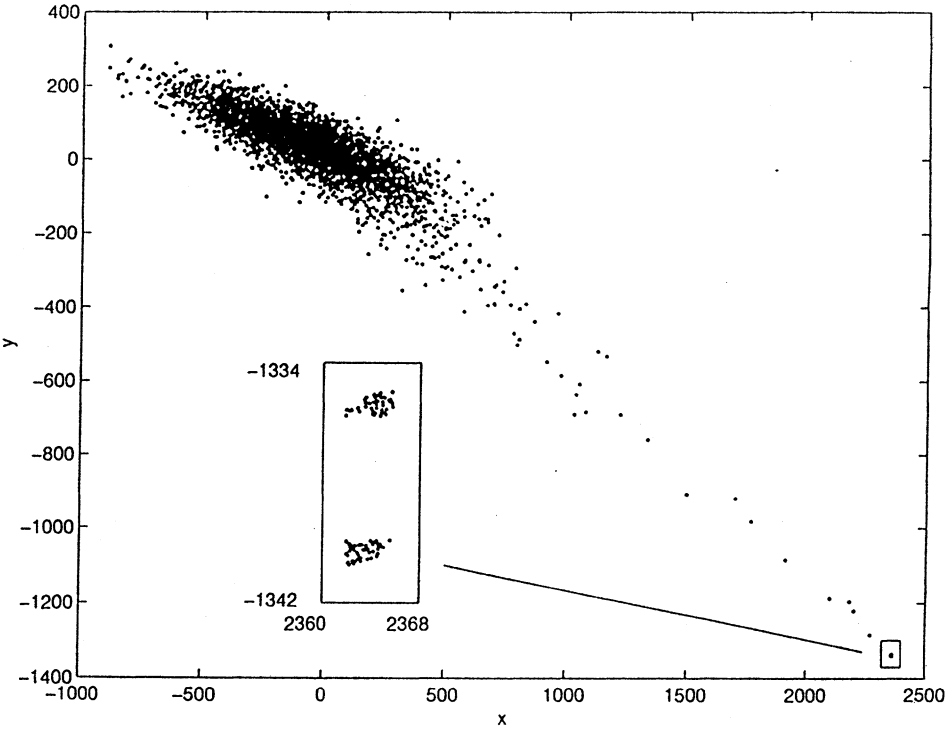

Figure 2: The dispersion of tracer particles, initially uniformly distributed in the $(2\pi)^2$ box in the phase-space after 500 modulation periods. The triad is the same as in Fig. 1.

amount of anomalous diffusion of tracer particles (Fig. 2). Although by technical reasons we did not explore the whole of parameter space and, thus, are unable to estimate a measure of islands-bearing triads we, nevertheless, see that these triads and, hence, the phenomenon of the ballistic transport are common. At the same time, a chaotic mixing related to the chaotic part of the phase portrait is universal. It should be noted that elliptic islands appear when nonlinearity is well-developed (the ratio $\max(T_i)/T_M \sim \frac{1}{3}$ or greater) while the threshold for chaotic mixing corresponds to higher values of β.

Thus, we have shown that chaotic mixing accompanied in many cases by ballistic advection of a fraction of tracer particles characterize the Lagrangian transport by resonant triads of Rossby waves. The fact that the nontrivial dynamical behavior we observed takes place in the domain of parameters corresponding to the transition from wave-dominated to vortex-dominated regimes in β-plane turbulence [1] suggests that this crossover regime may have nontrivial transport properties. This remains to be checked in direct numerical simulations.

References

[1] Rhines, P. B. (1975) "Waves and turbulence on a beta-plane" J. Fluid Mech. **69**, 417–443.

[2] Hirshman, S. P. (1980), "Two-dimensional electrostatic $E \times B$ trapping," Phys. Fluids **23**, 562–565.

[3] Pierrehumbert, R. T. (1991) "Chaotic mixing of tracer and vorticity by modulated travelling Rossby waves", Geoph. Astrophys. Fluid. Dyn. **58**, 285–319.

[4] Pierrehumbert, R. T. (1991) "Large-scale horizontal mixing in planetary atmospheres", Phys. Fluids A **3**, 1250–1260.

[5] del Castillo-Negrete, D. and Morrison, P. J. (1993) "Chaotic transport by Rossby waves in shear flow", Phys. Fluids A **5**, 948–965.

[6] Crisanti, A., Falcioni, M. Paladin, G. and Vulpiani, A. (1991) "Lagrangian chaos, transport, mixing and diffusion in fluids" Riv. Nuovo Cim. **12** #12, p. 1–80.

[7] Ottaviani, M. (1992) "Scaling laws of test particle transport in two-dimensional turbulence" Europhys. Lett. **20**, 11–116.

[8] Longuet-Higgins, M. S. and Gill, A. E. (1967) "Resonant interactions between planetary waves" Proc. Roy. Soc. Lond. A **299**(1456), 120–140.

[9] McLachlan, R. I. (1995), "On the numerical integration of ordinary differential equations by symmetric composition methods", SIAM J. Sci. Comput. **16**, 151–168.