Conventional calculations of the future of the Solar System quickly degenerate into disarray as computer errors build up. Symplectic integration could save the day

The world of

Robert McLachlan

WHAT will the Solar System be like in the distant future? Will Pluto and Neptune collide? Will the Earth be thrown into a different orbit by the combined gravitational pull from all the other planets? You might think that the answers are easily calculated. Just program a computer with Newton's laws of motion, tell it the positions of the planets now, and wait while it grinds out the future of the Solar System for the next billion years. Right?

Wrong. With a calculation as complicated as this, the computer is almost certain to come up with the wrong answer. It is not that the computer, or even that the person programming it, makes mistakes. The problem arises because computer replaces real time with a series of snapshots. Consequently, the calculations a computer makes are not absolutely precise, so it can only provide us with an approximate picture of what will happen in the real world. Normally the errors are so small that they go unnoticed, but when computers are set to work on the enormously long string of calculations needed to simulate the movement of the planets round the Sun, tiny errors in each step can build up to make the final result wildly inaccurate.

Confusion reigns

The errors are inevitable because the equations describing the Solar System are so complicated that precise solutions cannot even be attempted. Problems arise when errors build up systematically or, worse, when the errors become chaotic. If this happens, the error in the calculated result will not only be large, but unpredictable too. They can lead to results that defy the laws of physics—in an extreme case the planets could spiral into the Sun, for example, or gain energy from nowhere and spin off into space.

These errors affect every computer model although their effects go unnoticed because calculations usually run for a short time preventing wild variations from building up. Even a simple pendulum could start swinging like a propeller if the simulation were left to run for long enough.

Mathematicians have discovered they can get round these problems if the computer model is built not on the laws of motion that

'Movement in three-dimensional space is pure geometry in symplectic space'

Illustrations: Andrew Kingham

symplectic space

apply in our familiar three-dimensional space, but on the geometrical laws of a much larger mathematical world called symplectic space. What we understand as movement in our space can be represented as pure geometry in the very different world of symplectic space. This geometry provides a much more efficient way of representing movement mathematically: while it cannot prevent the computer from introducing errors, it ensures that, whatever the errors, the outcome is a physically reasonable one.

The laws of geometry in symplectic space are applied through mathematical tools known as symplectic integrators: simple formulae that a computer can use to create reliable simulations of chaotic and complex aspects of real systems. Symplectic integration is already helping scientists to model the forces between tens of thousands of atoms in a crystal lattice—a system far too complex for conventional methods to handle reliably—and successfully predict properties of the material such as its strength or the way it vibrates.

Not every system in our Universe is symplectic, however. Dissipative forces such as friction or viscosity do not obey geometric laws and cannot be simulated using symplectic integration. Applying the method to weather forecasting is

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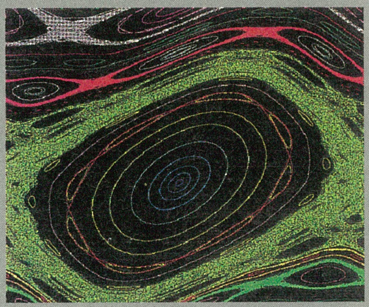
Symplectic map-making

The formulae used in symplectic calculations are very simple, the simplest of all being the "standard map". A "map" is simply mathematical jargon for a rule that shows how to move from one point to another. A computer given the coordinates of a point can use the standard map to generate a new point.



If the first point has coordinates x and y, the formula for generating the new coordinates is:

x_{new} = decimal part of $(x + y)$

$y_{new} = y - \sin(2\pi x_{new})$,

where "decimal part" means the numbers to the right of the decimal point.

As successive points are plotted on the same plane, a picture gradually builds up. Some starting points lead to a regular, repetitive cycle of points which eventually fill out to form a smooth curve. For others, the point bounces around seemingly at random. On the map shown above, different starting points are represented by different colours. Researchers have repeated these calculations with different starting points over a million billion (10^{15}) times to find the hidden patterns in this symplectic chaos.

The standard map turns out to be a mathematical model of a simple forced pendulum. Such models do not reproduce all the aspects of a real system but distil certain features that can then be studied. For example, the motion of an electron travelling round a circular particle accelerator can be described by the same model. In this case, the axes of the picture show the position and velocity of the particle. Different starting positions correspond to particles in different orbits, and the points on the picture are snapshots of the velocity and position of the particle at regular intervals. The orbits that remain stable produce a repetitive cycle of points, while chaotic orbits produce a random selection of points. Physicists use simulations like this when designing particle accelerators to determine how the beam will behave and how it must be controlled.

complicated, for example, because air resistance is a dissipative force and has to be ignored.

One example where symplectic simulations can help is in designing circular particle accelerators. The Main Ring accelerator at Fermilab in Illinois, which was built in 1972 before symplectic motion was understood, cannot store particles for long because small variations in their paths rapidly build up into uncontrollably large deviations. Physicists now realise that with the new symplectic methods they would have been able to simulate these effects, and design the machine to cope.

Conventional computer models use position and time to deduce velocity, but position and velocity are treated on an equal footing in symplectic space. For instance, in a model of the Solar System each planet is defined by six dimensions—three for its velocity in each direction and three for its position. The dimensions are coupled by special kinds of angles known as symplectic angles. These angles cannot be measured with a protractor: two lines superimposed on each other, for example, have a normal angle between them of 0° but a symplectic angle of 90°.

The special features of this weird space are its laws of geometry. That space and geometry are closely linked can be seen by looking at the properties of a triangle in two different types of space. We are taught at school that the angles of a triangle add up to 180°, but this is not always true if the triangle is not on a flat plane. For example, imagine a triangle on the surface of the globe, which has one corner at the North Pole and the other two on the equator. No matter what the angle at the pole, both the angles at the equator will be exactly 90°, so the three angles are bound to add up to more than 180°. By choosing the space carefully, mathematicians can arrange for certain geometric laws to hold true. In the example of the triangle, the angles add up to 180° only if the space is flat.

If the space is complex, then the geometric laws can be complex too. As the planets move in symplectic space according to the symplectic laws of geometry, so they move in three-dimensional space according to Newton's laws of motion. Symplectic geometry ensures that symplectic angles remain the same as the planets move. This geometry can describe all the complicated motion of the Solar System.

Symplectic space has been hard to explore because its geometry is so unlike that of three-dimensional space (the name comes from the Greek word *symplegma*, meaning tangled or plaited). After many decades of study, the breakthrough came during the 1960s when the Russian mathematician Vladimir Arnol'd at the Moscow State University, along with Andrei Kolmogorov and Jürgen Moser, proved a theorem that explains some of the implications that these hidden geometrical laws hold for real motion.

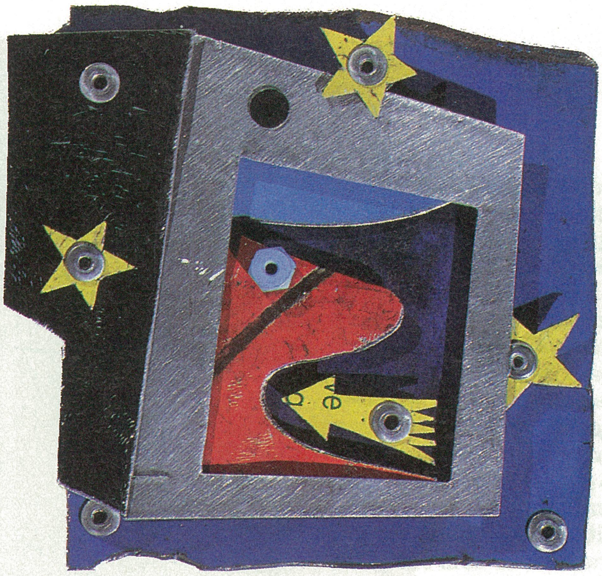
'The tilt of Pluto's orbit relative to the Earth's oscillates chaotically like a spinning coin coming to rest'

In the case of a single planet in a circular orbit around the Sun, the motion is nonchaotic and well understood. But one problem that could not be by conventional means is what happens when there is a second planet exerting a small gravitational pull on the first. Does the circular orbit merely become slightly elliptical? Does it develop a chaotic wobble? Or will the first planet wander off entirely? The Kolmogorov-Arnol'd-Moser (KAM) team proved that all three are possible. Given certain initial conditions, the orbit can still be regular. But if the starting conditions are slightly different, it could be chaotic. Once in a chaotic state, the orbit might even "leak out" in a process known as Arnol'd diffusion, which causes the planet to wander away from its circular orbit.

Alternative orbits

The problem lies in determining which type of orbit a system will adopt. The KAM theorem shows that chaos and order are infinitely mixed. Between any two regular orbits lie chaotic ones, and the planet could adopt any one of an infinite number of each type of orbit. But a planet that behaves nonchaotically can never become chaotic.

Traditional computer models of planetary orbits produce outrageous results because the build-up of errors in the calculation leads to results that run counter to the laws of motion on which the model is based. Symplectic integration avoids these pitfalls by modelling not just the



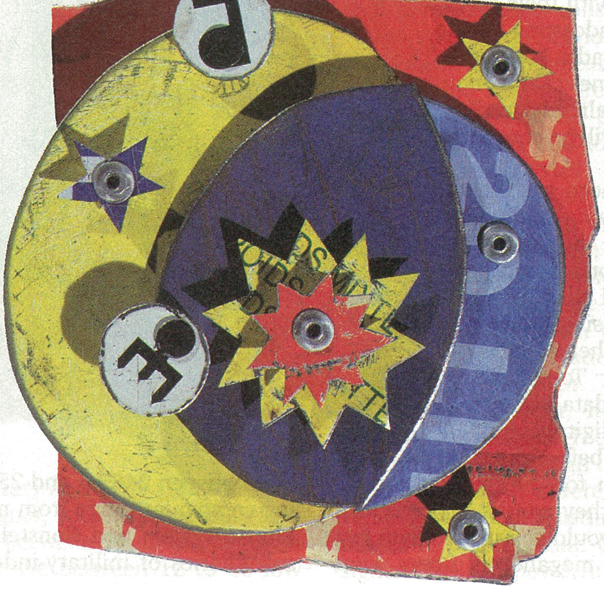
forces and accelerations as happens in conventional computer simulations, but by also keeping symplectic angles fixed in symplectic space. While the computation will still, inevitably, accumulate errors, the KAM theorem guarantees that they will not nudge a planet into an impossible orbit. The result does not predict the exact motion of our Solar System, but it does provide useful information about it. For example, astronomers can work out Pluto's distance from the Sun in a billion years' time, but not which side of the Sun it will then be on.

Although scientists have started to use symplectic integration only recently, it is not a new idea. In the 1950s René de Vogalaére, a mathematician then at the University of Notre Dame in Indiana, suggested rewriting formulae to preserve symplectic angles at each step. But his paper was rejected by a mathematical journal and the idea was forgotten. In 1983, it was rediscovered independently by Ronald Ruth at Lawrence Livermore National Laboratory in California and Feng Kang at the Chinese Academy of Sciences in Peking.

'Will Pluto collide with Neptune in a billion years' time? Surprisingly, the calculations clearly show the planets avoiding each other—a phenomenon that has yet to be explained'

Since then, the study of symplectic integration has gone from strength to strength and the technique is giving scientists new insights into the workings of the Solar System. That chaotic motion exists in the Solar System was first suggested in 1988 by conventional computer calculations. But these were painfully and unnecessarily slow, and symplectic integrations can be carried out in a fraction of the time to simulate this chaos far into the future. Scientists can now model the entire lifespan of the Solar System, thought to be about 10 billion years, and know that the model is reliable because it obeys the laws of symplectic geometry.

These models demonstrate vividly how chaotic the Solar System's behaviour is. Imagine a model of the Solar System made up of little coloured lights zipping around the Sun in elliptical orbits. Now speed up time until the points of light appear to be smeared into nine elliptical rings, with Neptune's orbit overlapping Pluto's. The first thing you notice is that the planets' perihelia—the points at which their elliptical orbits are closest to the Sun—are not fixed, but rotate slowly. This phenomenon, which is known as perihelion precession, has been known for around a century. The Earth's ellipse, for example, takes 112 000 years to rotate once—the sort of period that conventional computer methods can easily simulate.



Now speed up time even more, so that a million years pass every second. At this rate, Pluto orbits the Sun 4000 times each second and conventional computer methods start to fail within a few seconds. The measure of the shape of a planet's orbit is its eccentricity—the narrower the ellipse, the greater the eccentricity—and Pluto has the most eccentric orbit of all the planets. Symplectic integration can show how the cumulative effects of the gravitational pull of each planet on every other planet begin to show up in changes in the shape of their orbits. The eccentricity of Pluto's orbit begins to change erratically over the next 70 million years, leading to a 500-million-kilometre change in its maximum distance from the Sun.

Pluto's perihelion also behaves strangely. Instead of steadily rotating, as other planets' perihelia do, it comes to a halt, reverses, and finally swings back and forth by an amount that varies every 34 million years. In addition, the tilt of its orbit relative to the Earth's, oscillates chaotically between 14° and 17°, like a spinning coin coming to rest. Jack Wisdom of the Massachusetts Institute of Technology, who calculated these figures, says the diversity of Pluto's motion seems to be inexhaustible.

So will Pluto collide with Neptune in a billion years' time? Probably not. Surprisingly, the calculations clearly show the planets avoiding each other—a phenomenon that has yet to be explained. The Solar System may behave chaotically, but it seems destined never to look very different from the way it is now. □

Robert McLachlan *is an applied mathematician at Massey University, New Zealand.* **Further reading** *"The symplectic revolution: A new suppler geometry is transforming physics and mathematics",* The Sciences, *vol 30 p 28.* Newton's Clock: Chaos in the Solar System, *by Ivars Peterson, W. H. Freeman 1993*