On a possible mechanism of anomalous diffusion by Rossby waves

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We study the advection of passive tracers by traveling plane Rossby waves of finite amplitude. In distinction with previous studies the nonlinearity of the wave field is taken into account in the first order of perturbation theory by considering the Lagrangian transport by resonant wave triads. Using the waves' phases as new dynamical variables we reduce the problem to the study of a specific one-and-a-half degree of freedom Hamiltonian system with nonharmonic modulation. By using a symplectic integrator we study this system numerically and find an interesting series of bifurcations of its phase portrait as the nonlinearity increases. As is standard in the systems of this type we commonly see a chaotic sea with elliptic islands in the phase space, which means that in the physical space the resonant triads give rise to chaotic mixing and ballistic transport, respectively. The relevance of these results to the transport properties of β-plane turbulence is discussed. © *1998 American Institute of Physics.* [S1070-6631(98)02712-3]

I. INTRODUCTION

Rossby waves characterize the response of a differentially rotating fluid layer to small perturbations and, hence, represent one of the essential ingredients of large-scale quasi-two-dimensional atmosphere/ocean dynamics. Being constantly present in the atmosphere and oceans these waves influence tracer transport and, thus, a question arises about the behavior of Lagrangian particles (transporting the tracer) advected by a system of Rossby waves. The particle–wave(s) interaction is traditionally studied in the plasma physics example of charged particles in an electromagnetic field, and has become one of the archetypes of modern dynamical systems theory (see, e.g., Refs. 1, 2). Much is known about the transport and mixing properties of this system (cf. Refs. 3, 4). A natural extension of these results to geophysical fluid dynamics problems was undertaken first in Ref. 5, where advection of a passive tracer by a pair of Rossby waves propagating along a channel [i.e., of the form $\sin k(x-ct)\sin ly$] was investigated. The phenomenon of chaotic mixing was displayed and applications to observed atmospheric flows were considered.[6] Chaotic transport by Rossby waves in a shear flow was later studied in Ref. 7.

It should be stressed, however, that a crucial difference exists between the behavior of charged particles in an electromagnetic field and that of Lagrangian fluid particles. Normally, the dynamics of the former becomes nontrivial once a certain threshold in wave amplitudes is passed. But, while the nonlinear interaction of electromagnetic waves is weak, even for large wave amplitudes and, usually, may be safely neglected, as well as the back-reaction of the particle motion on the wave field, this is not the case for Rossby waves (and other waves in fluids). In addition, unlike the case of charged particles in electrodynamics, the motion of the Lagrangian fluid particles should satisfy specific global constraints such as the conservation of circulation along a moving liquid contour. These constraints have a drastic influence on the character of the motion.[8] Thus, a problem of self-consistency of the kinematic description where the Lagrangian fluid particles are advected by a given wave configuration arises if this latter *is not* an exact solution of the wave-field evolution equation. Although the authors of Refs. 5–7 were perfectly aware of this fact, they limited their study to the kinematic aspects of the problem, discarding the nonlinear wave interaction even for relatively large wave amplitudes.

It should be emphasized that solving the above-mentioned self-consistency problem for the Lagrangian transport by a flow evolving from an initial state consisting of a system of finite-amplitude Rossby waves would mean solving the problem of beta-plane turbulence, which is, obviously, impossible by analytical means. We may, however, make a step forward toward realistic dynamics by introducing perturbatively the effects of nonlinear wave interactions and, thus, pushing the inconsistency to longer times (of the order ϵ^{-1} in nondimensional units, where ϵ is the nondimensional wave amplitude). Our main focus will be on the ballistic transport by the traveling plane Rossby waves—an effect analogous to the electrostatic particle trapping in plasma first discussed in Ref. 9. As was just said, instead of considering the advection by a pair of harmonic waves we take into account the fact that in the leading order of the perturbation theory in wave amplitude the two initial waves will generate

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a third one forming a resonant triad and all three will evolve in time subject to a slow nonharmonic modulation. It is the advection by such resonant wave triads which we are interested in that follows. The velocity field produced by resonant triads is spatially and time periodic. General conditions for anomalous transport in such systems were analyzed in Ref. 10, where it was shown that the flow should be not uniformly ergodic to be able to support ballistic advection. Below we show that it is indeed the case for Rossby wave triads and that they do ballistically transport tracers.

II. PRELIMINARIES: ROSSBY WAVES DYNAMICS, AND RESONANT TRIADS

Below, we shall limit ourselves to one of the simplest systems exhibiting Rossby waves, namely, a purely two-dimensional incompressible fluid motion in the plane tangent to the rotating planet surface (i.e., β-plane motion). The generalizations to more realistic problems (quasigeostrophic barotropic motion with a finite deformation radius, multilayered or continuously stratified models, cf. Ref. 11, or spherical geometry, cf. Ref. 12) are straightforward. Thus, the equation for the evolution of the streamfunction ψ of the two-dimensional velocity field $\mathbf{v}=(-\partial\psi/\partial y, \partial\psi/\partial x)$ is

$$\frac{\partial\Delta\psi}{\partial t}+J(\psi,\Delta\psi)+\beta\,\frac{\partial\psi}{\partial x}=0, \tag{1}$$

where β is the Coriolis parameter, $J(A,B)=(\partial A/\partial x)\times(\partial B/\partial y)-(\partial A/\partial y)(\partial B/\partial x)$, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ and $\mathbf{x}=(x,y)$ are Cartesian coordinates on the β plane. The linear part of this equation describes propagation of Rossby waves of the form $\psi=a\cos(\omega t-\mathbf{k}\cdot\mathbf{x}+\varphi)$ with dispersion relation

$$\omega=\Omega(k_x,k_y)=-\beta\,\frac{k_x}{\mathbf{k}^2}. \tag{2}$$

Let us note in passing that a single harmonic wave of arbitrary amplitude is an exact solution of (1). As is well known (cf. e.g., the classical paper, Ref. 13 or 11), in the lowest order of the perturbation expansion in wave amplitude the Rossby waves' interactions are dominated by resonant triads. Let us be reminded that the resonant interactions appear naturally in the standard ''two-timing'' perturbative technique in the modulation theory (see, e.g., Ref. 14). The classical three-wave equations arise in the simplest version of this technique where a slow time dependence is introduced only in the amplitudes of the basic harmonic waves, cf. Refs. 13, 15, as it is sketched below (a generalization of our results for the case of the full modulation equations including slow time dependence of the phases, i.e., spatial modulation, cf., e.g., Ref. 16, is also possible but will be not discussed in what follows).

Indeed, if one takes a triad of waves,

$$\psi_1=\sum_{i=1}^{3} a_i\cos\Phi_i, \tag{3}$$

satisfying the resonance conditions

$$\Phi_1+\Phi_2=\Phi_3,\quad \forall\mathbf{x},t, \tag{4}$$

where

$$\Phi_i=\Omega_i t-\mathbf{k}_i\cdot\mathbf{x}, \tag{5}$$

$\Omega_i=\Omega(k_{x_i},k_{y_i})$, ϵ is the small parameter, and $a_i=a_i(\tau)$ where $\tau=\epsilon^{-1}t$ is the slow modulation time, as a first-order approximation to the regular perturbative expansion

$$\psi=\epsilon\psi_1+\epsilon^2\psi_2+\cdots,$$

of the solution of (1); then the integrability conditions for the next approximation, ψ_2, give modulation equations for the wave amplitudes,

$$\frac{\partial a_1}{\partial\tau}=\gamma_1 a_2 a_3,\qquad \frac{\partial a_2}{\partial\tau}=\gamma_2 a_3 a_1,\qquad \frac{\partial a_3}{\partial\tau}=\gamma_3 a_1 a_2, \tag{6}$$

where γ_i, $i=1,2,3$ are the interaction parameters,

$$\gamma_1=\frac{\mathbf{k}_2\times\mathbf{k}_3}{\mathbf{k}_1^2},\quad \gamma_2=\frac{\mathbf{k}_3\times\mathbf{k}_1}{\mathbf{k}_2^2},\quad \gamma_3=\frac{\mathbf{k}_1\times\mathbf{k}_2}{\mathbf{k}_3^2}, \tag{7}$$

and the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a}\times\mathbf{b}=a_x b_y-a_y b_x$. Note that these equations result from (6.1) of Ref. 13 after a change $\mathbf{k}_i^2+1\rightarrow\mathbf{k}_i^2$ which corresponds to our choice (1) of the nondivergent planetary wave equation, in the terminology of Ref. 13. The solution of (6) may be written in terms of elliptic functions (see, e.g., Ref. 17):

$$a_1(\tau)=a_{1_0}\,\mathrm{dn}(\tau|\kappa),\quad a_2(\tau)=a_{2_0}\,\mathrm{sn}(\tau|\kappa),\quad a_3(\tau)=a_{3_0}\,\mathrm{cn}(\tau|\kappa), \tag{8}$$

where, without loss of generality, we have assumed that $a_2(0)=0$. In terms of the initial amplitudes and interaction parameters, the modulus κ of the elliptic functions is given by

$$\kappa^2=\frac{a_{3_0}^2}{a_{1_0}^2}\left|\frac{\gamma_1}{\gamma_2}\right|. \tag{9}$$

The requirement $\kappa<1$ constrains the maximum value of a_{3_0}/a_{1_0}, this maximum depending on the triad. The modulation period, which we call T_M, is equal to the period $K(\kappa)$ of the elliptic functions, where K is the complete elliptic integral of the first kind. We should emphasize at this point that (3), (4), (8) do not specify an exact solution of (1), although we do have that the initial energy and enstrophy of a resonant triad are exactly conserved.[13] Indeed, corrections containing combinational frequencies are necessarily generated, but they are small as the initial wave amplitudes are small. One may take the smallness of the ratio

$$\delta=\frac{\max(T_i)}{T_M}=\frac{\max 2\pi/\Omega_i}{K(\kappa)}, \tag{10}$$

where $T_i=2\pi/\Omega_i$ are the oscillation periods of the triad components, as a rough criterion of applicability of the resonant triad description of the wave field generated by an initially active pair of waves 1 and 3 [cf. (8)] in a nondegenerate case $\Omega_i \neq 0$. (See the Figure Caption below for the values of δ in numerical calculations.)

The following explicit parametrization of resonant triads satisfying (4) will be used below. Consider a triangle formed by the resonant wave vectors $\mathbf{k}_i$, $i=1,2,3$ and denote by α the angle formed by $\mathbf{k}_1$ and $\mathbf{k}_3$ and by γ that formed by $\mathbf{k}_2$ and $\mathbf{k}_3$. The resonant conditions (4) are equivalent to

$$\Omega_1+\Omega_2=\Omega_3, \quad \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3, \tag{11}$$

and are invariant with respect to dilation of the wave vectors [cf. (2)]. Hence, without loss of generality we may choose $|\mathbf{k}_3|=1$ and rewrite (11) in terms of the wave-vectors moduli and polar angles as

$$\begin{aligned} \cos\varphi_3 &= |\mathbf{k}_1|\cos\varphi_1+|\mathbf{k}_2|\cos\varphi_2, \\ \sin\varphi_3 &= |\mathbf{k}_1|\sin\varphi_1+|\mathbf{k}_2|\sin\varphi_2, \\ \cos\varphi_3 &= |\mathbf{k}_1|^{-1}\cos\varphi_1+|\mathbf{k}_2|^{-1}\cos\varphi_2. \end{aligned} \tag{12}$$

Then elementary trigonometry shows that

$$\varphi_1=\varphi_3+\alpha, \quad \varphi_2=\gamma-\varphi_3,$$

$$\varphi_3=\arctan\left(\frac{\dfrac{1}{\sin(\alpha+\gamma)}-\dfrac{\cos\alpha}{\sin\gamma}-\dfrac{\cos\gamma}{\sin\alpha}}{\dfrac{\sin\gamma}{\sin\alpha}-\dfrac{\sin\alpha}{\sin\gamma}}\right) \tag{13}$$

and

$$|\mathbf{k}_1|=\frac{\sin\gamma}{\sin(\alpha+\gamma)}, \quad |\mathbf{k}_2|=\frac{\sin\alpha}{\sin(\alpha+\gamma)}. \tag{14}$$

Thus triad space corresponds to the triangle,

$$\{(\alpha,\gamma):0\leqslant\alpha\leqslant\pi,\gamma\leqslant\pi,\alpha+\gamma\leqslant\pi\}. \tag{15}$$

We are particularly interested in triads that allow a wide range of physically consistent waves those with $\delta\ll 1$ from (10). Of course, $\delta\to 0$ as $\beta\to\infty$ and as $\kappa\to 1$. However, we shall see that the interesting, anomalous transport modes arise for *small* values of β; and as $\kappa\to 1$, the amplitudes $a_i(t)$ tend to periodic step functions, whose large gradients also destroy the modes we wish to study. Thus, we seek those triads that minimize β max $T_i=2\pi|\mathbf{k}|_i/k_{x_i}$, to allow the greatest range of β and a_{3_0} while retaining $\delta<1$. The global minimum occurs at $\gamma_*=0.66$, $\alpha_*=0.97$.

There is one family of triads that seem particularly interesting in their own right. These are the isosceles triads with two values of $|\mathbf{k}|$ equal. There are three such families: with $\alpha=\gamma$, $\alpha=\pi-\alpha-\gamma$, and $\gamma=\pi-\alpha-\gamma$. In these cases $\kappa=0$ and the elliptic functions (8) reduce to the trigonometric functions $a_1=a_{1_0}$, $a_2=a_{2_0}\sin(\tau)$, $a_3=a_{3_0}=\cos(\tau)$. However, this case is degenerate in the sense that the third wave vector is always vertical: it has $k_x=0$, and hence its wave has frequency 0 representing a plane-parallel (co)sinusoidal flow on the β plane (a Kolmogorov flow). Thus, the triads in question should be relevant, e.g., in studies of stability of this flow.

III. LAGRANGIAN TRANSPORT BY ROSSBY WAVES

Lagrangian fluid particles advected by the flow obey the equations of motion,

$$\dot{x}=-\frac{\partial\psi}{\partial y}, \quad \dot{y}=\frac{\partial\psi}{\partial x}, \tag{16}$$

where $\mathbf{x}(t)=[x(t),y(t)]$ is a particle position on the plane at time t. These equations represent a ''one-and-a-half'' degree of freedom Hamiltonian system that may be studied by standard methods once ψ, the Hamiltonian, is known as a function of x, y, and t.

Before considering advection by a resonant triad, we first summarize the dynamics of (16) in the presence of only one or two harmonic waves.

For a single propagating harmonic wave (16) is easily integrated and gives simple oscillations of particles' positions. For a pair of propagating harmonic waves,

$$\psi=A_1\cos(\Omega_1 t-\mathbf{k}_1\cdot\mathbf{x})+A_2\cos(\Omega_2 t-\mathbf{k}_2\cdot\mathbf{x}), \tag{17}$$

Eqs. (16) are also integrable, as the time dependence may be eliminated by choosing the phases $\Phi_{1,2}$ as new dynamical variables. [Note the difference between this case and the nonintegrable case,

$$\begin{aligned}\psi=&A_1\sin k_1(x-c_1t)\sin l_1y\\ &+A_2\sin k_2(x-c_2t)\sin l_2y,\end{aligned}$$

treated in Ref. 5]. These variables are defined on the torus $\mathbf{T}^2$ but may be lifted to its universal cover $\mathbf{R}^2$. The change of variables $\mathbf{x}\mapsto\mathbf{\Phi}$ is a good one if the matrix

$$K=\begin{pmatrix} k_{1_x} & k_{1_y} \\ k_{2_x} & k_{2_y}\end{pmatrix} \tag{18}$$

is nonsingular. This is true exactly in the interior of the triad triangle (15). In this case we get a system of equations,

$$\begin{aligned}\dot{\Phi}_1&=\Omega_1+(\mathbf{k}_1\times\mathbf{k}_2)A_2\sin\Phi_2,\\ \dot{\Phi}_2&=\Omega_2-(\mathbf{k}_1\times\mathbf{k}_2)A_1\sin\Phi_1,\end{aligned} \tag{19}$$

which is an integrable canonical one-degree-of-freedom Hamiltonian system with Hamiltonian

$$H=\Omega_1\Phi_2-\Omega_2\Phi_1-(\mathbf{k}_1\times\mathbf{k}_2)(A_1\cos\Phi_1+A_2\cos\Phi_2). \tag{20}$$

We call $\mathbf{x}$ *physical space*, and $\mathbf{\Phi}=(\Phi_1,\Phi_2)$ *phase space*. They are related by

$$\mathbf{x}=(K^{-1}\mathbf{\Omega})t-\mathbf{\Phi}. \tag{21}$$

If a particle has $\mathbf{x}(t)$ bounded we call it *frozen*; if it is not bounded, but $\lim_{t\to\infty}\mathbf{x}(t)/t=0$, we say it is *diffusive* [and, generally, $\mathbf{x}(t)\sim t^{1/2}$]; if a nonzero $\lim_{t\to\infty}\mathbf{x}(t)/t$ exists, we say the motion is *ballistic*. Ballistic motion corresponds to particles being advected with, on average, a constant velocity; we call these particles *shooters*. In general, the situation where $\mathbf{x}(t)\sim t^{\alpha}$ with $\alpha\neq\frac{1}{2}$ is known as anomalous diffusion.

For advection by a pair of harmonic waves [Eqs. (17), (19), (20)], the motion in phase space is along the isoenergetic curves $H=$const with an average (over $\mathbf{\Phi}\in\mathbf{T}^2$) veloc-

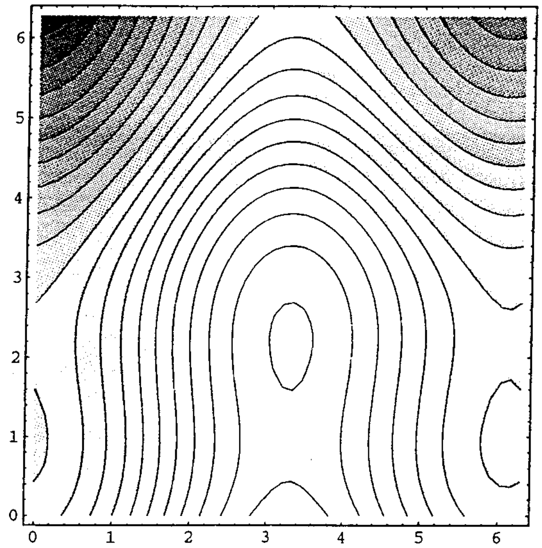

FIG. 1. The contours of $H=\mathrm{const}$ in the (Φ_1,Φ_2) plane for a pair of waves $\mathbf{k}_1=(1,1)$; $\mathbf{k}_2=(1,2)$ with $A_{1_0}=1$, $A_{2_0}=0.5$, $\beta=0.8$.

ity $\mathbf{\Omega}=(\Omega_1,\Omega_2)$, which gives no average (over $\mathbf{\Phi}$) displacement in physical space. However, two different situations may take place according to the topology of the isoenergetic curves, which is determined by the interplay of the linear and nonlinear terms in (20). In the case when the linear term is dominating the isolines of H are open and each point in the phase space is moving with a velocity close to the average velocity. In physical space any particle therefore remains close to its initial position and is frozen. In the case when linear and nonlinear terms are comparable, fixed points, each with a surrounding island of periodic orbits (elliptic islands) appear in the phase space. (See Fig. 1.). A fixed point in $\mathbf{\Phi}$ space, by virtue of (5), corresponds to a ballistically advected particle in physical space. It is clear that the whole elliptic island around a given fixed point will be also ballistically advected. The relative number of shooters is, thus, defined by the fraction of phase space occupied by the elliptic islands. All other points moving along the open orbits in phase space stay close to the origin in the physical space; but, in order to maintain zero average velocity, the cloud of ''normal'' points must drift in the direction opposite to the shooters' motion.

The above-described ballistic advection phenomenon was discussed in plasma physics in Ref. 9. In the present context, if confirmed, it would imply interesting consequences as to the transport of passive scalars in geophysical flows. However, the condition for the appearance of shooters means that the nonlinearity is strong and, thus, the question of consistency described in the Introduction arises. We therefore make the next step and consider a resonant triad engendered by an initial pair of waves. That is, we consider a streamfunction of the form (3) with the amplitudes satisfying (8). The resonance condition (4) allows us to eliminate the third phase Φ_3 and, again, to get a Hamiltonian system defined in the phase space (Φ_1,Φ_2). The equations of motion and the Hamiltonian are, respectively,

$$\dot{\Phi}_1=\Omega_1+(\mathbf{k}_1\times\mathbf{k}_2)[a_2(t)\sin\Phi_2+a_3(t)\sin(\Phi_1+\Phi_2)],$$
$$\dot{\Phi}_2=\Omega_2-(\mathbf{k}_1\times\mathbf{k}_2)[a_1(t)\sin\Phi_1+a_3(t)\sin(\Phi_1+\Phi_2)],\tag{22}$$

$$H=\Omega_1\Phi_2-\Omega_2\Phi_1-(\mathbf{k}_1\times\mathbf{k}_2)[a_1(t)\cos\Phi_1+a_2(t)\times\cos\Phi_2+a_3(t)\cos(\Phi_1+\Phi_2)],\tag{23}$$

and represent a one-and-a-half degree of freedom dynamical system that is not integrable due to the explicit time dependence of the a_i. Note that the modulation $a_i(t)$, although periodic with period T_M, is not harmonic, except for the degenerate case of the isosceles triads [cf. (6), (7)]. A standard method of investigation for this type of dynamical system is a numerical integration and a study of the phase portrait resulting from the iterations of the Poincaré map f_{P} over the modulation period T_M. Below we present the results of such analysis for selected triads. It is difficult to explore systematically the whole parameter space of this model—while the triads' geometry is defined by γ and α [see (13), (14)], there are still two additional parameters in (22), namely, the amplitude ratio a_{3_0}/a_{1_0} (we may fix $a_{1_0}=1$ without a loss of generality) and β. Bearing in mind the consistency condition saying that the oscillation periods have to be much smaller than the modulation period, we have chosen to work in the vicinity of the point $\gamma_*=0.66$, $\alpha_*=0.97$, which provides a global minimum in triad space of the shortest oscillation period normalized by β.

In general, for the nonintegrable system (22) one might expect a chaotic sea with, possibly, some islands of regular behavior as a typical phase portrait. As $\beta\rightarrow\infty$ is an integrable limit of (22), one might also expect that for any triad a more and more regular pattern of dynamical behavior emerges with increasing β. As in the integrable case of a pair of waves, the presence of elliptic islands would signify a presence of shooters. Indeed, suppose the Poincaré map f_{P}: $\mathbf{T}^2\rightarrow\mathbf{T}^2$ of (22) has a period n fixed point $\mathbf{\Phi}_0$ with a winding number $\mathbf{m}$ about the torus, and n is the discrete time. This means that in terms of the lift map $\tilde{f}_{\mathrm{P}}$ on $\mathbf{R}^2$, we have

$$\tilde{f}^n_{\mathrm{P}}(\mathbf{\Phi}_0)=\mathbf{\Phi}_0+2\pi\mathbf{m},\tag{24}$$

and the distance traveled in time nT_M in physical space is

$$\mathbf{x}=K^{-1}\cdot(\mathbf{\Omega}nT_M-2\pi\mathbf{m})+\mathbf{x}(0).\tag{25}$$

Hence, the fixed points and the regular orbits around them belonging to the stability islands having the same average winding number correspond to shooters. Chaotic orbits correspond to slow diffusion. The presence of stability islands results in a slow drift in physical space of the diffusive cloud away from the shooters, so as to compensate for the fast escape of the shooters and to keep the average winding number zero.

The above-described scenario is an illustration of the general analysis of Ref. 10, where it was shown that a necessary condition of the ballistic advection in space and time periodic velocity fields is the nonergodicity (i.e., the presence of stability islands) of the flow.

IV. NUMERICAL PROCEDURE AND RESULTS

We wish to follow many orbits of (22) for long times at many different parameter values. By using a symplectic integrator,[18,19] we ensure that the computed Poincaré map f_P is exactly (up to roundoff error) area preserving and that no spurious non-Hamiltonian-like dynamics or bifurcations will be observed. As the system (22) is nonautonomous, it is clearest to derive the symplectic integrator in the extended phase space $(\Phi_1, \Phi_2, \xi, p_\xi)$ in which $\xi = t$. The extended Hamiltonian is $\tilde{H} = H + p_\xi$, and this Hamiltonian may he split into four pieces, each of which is integrable:

$$\tilde{H} = H_1 + H_2 + H_3 + H_4,$$

$$H_1 = \Omega_1 \Phi_2 - (\mathbf{k}_1 \times \mathbf{k}_2) a_2(t) \cos \Phi_2,$$

$$H_2 = -\Omega_2 \Phi_1 - (\mathbf{k}_1 \times \mathbf{k}_2) a_1(t) \cos \Phi_1,$$

$$H_3 = -(\mathbf{k}_1 \times \mathbf{k}_2) a_3(t) \cos(\Phi_1 + \Phi_2),$$

$$H_4 = p_\xi.$$

Writing the time steps as, e.g., $\Phi \mapsto \Phi'$, the time-Δt flows of these Hamiltonians are

$$H_1: \quad \Phi_1' = \Phi_1 + \Delta t [\Omega_1 + (\mathbf{k}_1 \times \mathbf{k}_2) a_2(\xi) \sin \Phi_2]$$

$$H_2: \quad \Phi_2' = \Phi_2 + \Delta t [\Omega_2 - (\mathbf{k}_1 \times \mathbf{k}_2) a_1(\xi) \sin \Phi_1]$$

$$H_3: \quad \Phi_1' = \Phi_1 + c, \quad \Phi_2' = \Phi_2 - c,$$

$$c = \Delta t (\mathbf{k}_1 \times \mathbf{k}_2) a_3(\xi) \sin(\Phi_1 + \Phi_2),$$

$$H_4: \quad \xi' = \xi + \Delta t.$$

For an algorithm with second-order accuracy, we compose these flows in the pattern 1, 2, 3, 4, 4, 3, 2, 1. Notice how the Hamiltonian splitting has the effect of ''freezing'' the explicit time dependence of H during the integration. To compute the Poincaré map, we take an integer number of time steps (usually 50 or 100) per period T_M of the forcing functions $a_i(t)$. As most orbits are chaotic and cannot be followed accurately anyway, we believe that second-order accuracy is sufficient, although we have, of course, checked that our results do not depend on the time step Δt.

We use the explicit parametrization of triads given by (13), (14). The elliptic functions defining the time dependence of a_i are also calculated numerically using a standard algorithm.[20]

A bifurcation sequence for the phase portrait of the triad with $\gamma = 0.7$, $\alpha = 1$ is represented in Figs. 2–6 for β decreasing from 2 to 0.5. The initial amplitude a_{3_0} was chosen to be 0.66 (quite close to its allowable maximum in this case of 0.67) in order to maximize T_M. The sequence is characterized by a regular ergodic (i.e., space filling but not mixing) behavior for large β, as expected (Fig. 2), which gives place to chaos (Fig. 3) by more and more vigorous distortion of the phase trajectories and, then, to the birth of multiple relatively small and/or narrow islands (Fig. 4). These primary islands disappear giving rise to chaos (Fig. 5) from which secondary ''fat'' islands are born (Fig. 6). It is these fat islands that give rise to a relatively large numbers of shooters and a corresponding large amount of anomalous diffusion of tracer particles. Interestingly, these islands do not arise from a resonance of the known $\beta \to \infty$ integrable limit. This makes them difficult to predict or account for analytically. (They may be remnants of some *other* integrable limit of the model, even if this limit is not reached through physically consistent values of the parameters.)

For reasons of physical consistency of the model we do not show the subsequent bifurcations that take place beyond the limit $T_i \sim T_M$, although they may be interesting in them-

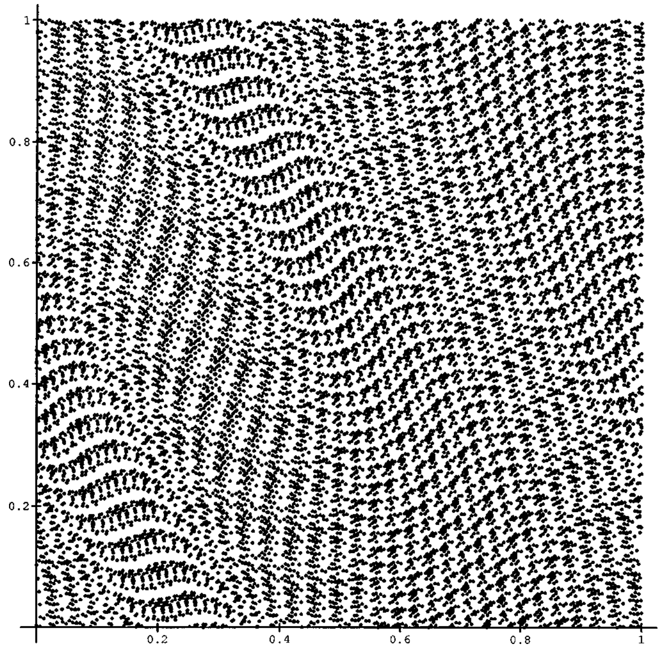

FIG. 2. The phase portrait representing ten different trajectories of the triad $\gamma = 0.7$, $\alpha = 1$ at $\beta = 2$ after 2000 T_M. The modulation period $T_M = 47.14$. The wave periods T_i are 5.66, 2.74, 5.32, respectively.

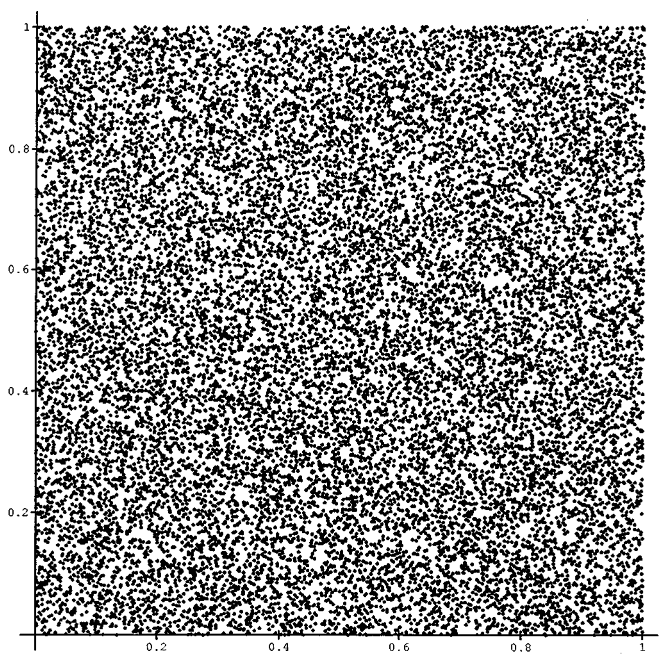

FIG. 3. The phase portrait of the triad $\gamma = 0.7$, $\alpha = 1$ at $\beta = 1$ after 2000 T_M; $T_M = 47.14$. The wave periods T_i are 11.33, 5.49, 10.64, respectively.

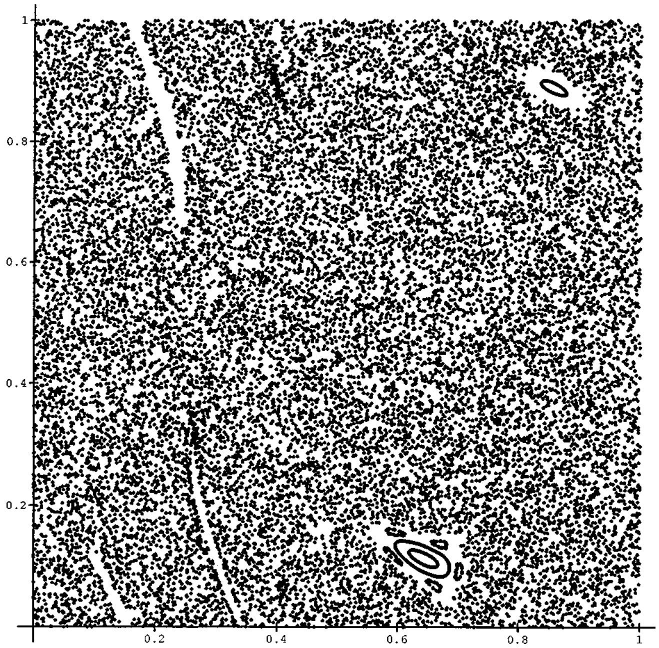

FIG. 4. The phase portrait of the triad $\gamma=0.7$, $\alpha=1$ at $\beta=0.75$ after 2000 T_M; $T_M=47.14$. The wave periods T_i are 15.11, 7.32, 14.19, respectively.

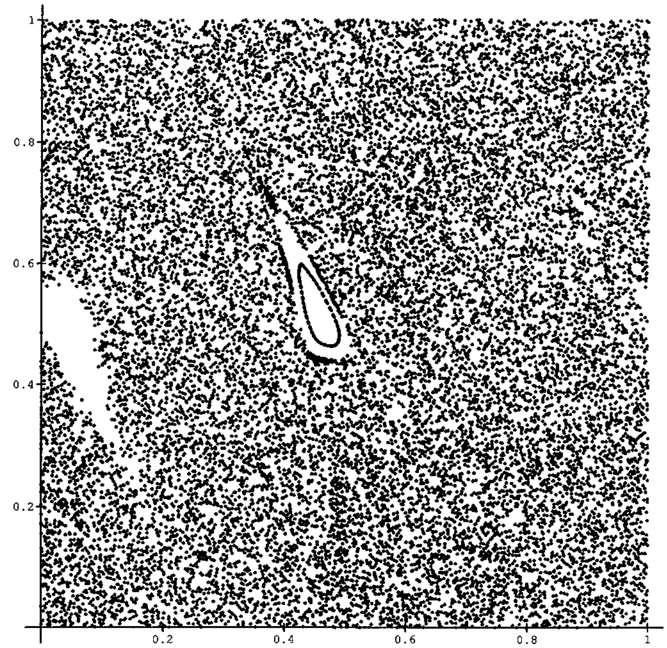

FIG. 6. The phase portrait of the triad $\gamma=0.7$, $\alpha=1$ at $\beta=0.5$ after 2000 T_M; $T_M=47.14$. The wave periods T_i are 22.66, 10.98, 21.27, respectively. An elliptic trajectory inside the island is also shown.

selves. A corresponding evolution of the cloud of tracer particles in the physical space is shown in Figs. 7–9. A similar scenario was observed for other triads close to γ_*, α_*, although the number of primary narrow islands varies. We also remark that the presence or absence of elliptic islands is sensitive to the ratio a_{3_0}/a_{1_0}. Another generic scenario consisting in a direct transition to quasiuniform chaos from the regular behavior while decreasing β was also widely observed. Although for the above-explained reasons we did not explore the whole of parameter space and, thus, are unable to estimate a measure of island-bearing (i.e., shooter-supporting) triads we, nevertheless, see that these triads and, hence, the phenomenon of the ballistic transport, are common. At the same time, a chaotic mixing related to the chaotic part of the phase portrait is universal. It should be noted that elliptic islands appear when nonlinearity is well devel-

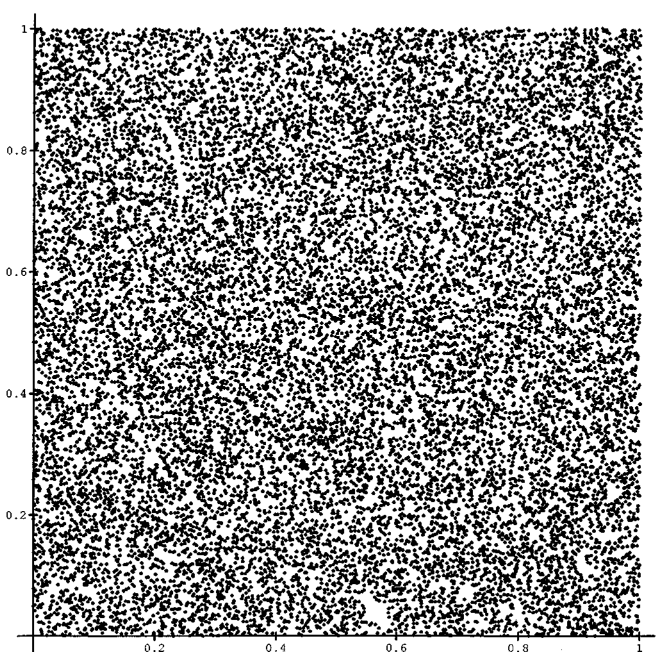

FIG. 5. The phase portrait of the triad $\gamma=0.7$, $\alpha=1$ at $\beta=0.7$ after 2000 T_M; $T_M=47.14$. The wave periods T_i are 16.18, 7.84, 15.21, respectively.

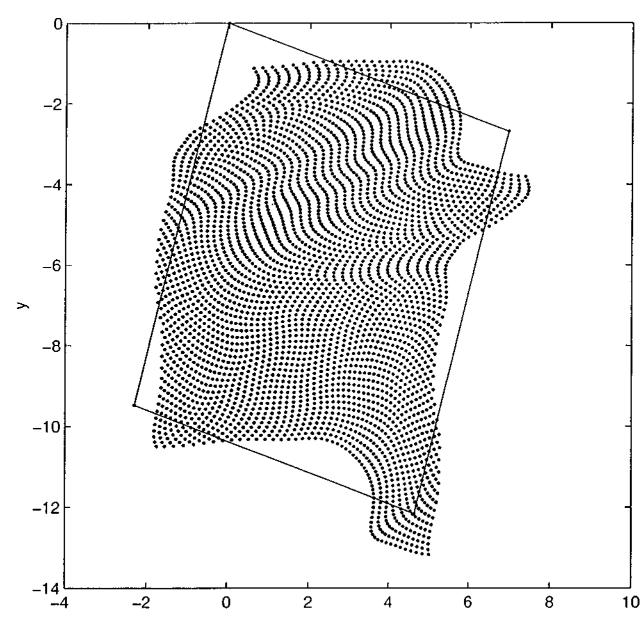

FIG. 7. The dispersion of tracer particles, initially uniformly distributed in the a $(2\pi)^2$ box in the phase space after 500 modulation periods; the initial box is also shown. The triad is the same as in Fig. 2.

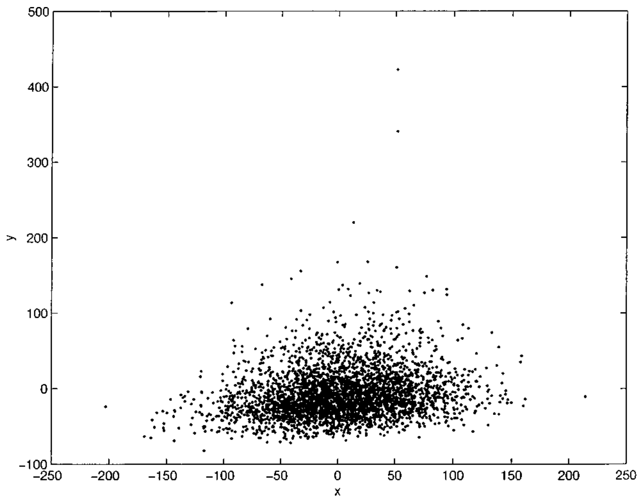

FIG. 8. The dispersion of tracer particles, initially uniformly distributed in the $(2\pi)^2$ box in the phase space after 500 modulation periods. The triad is the same as in Fig. 5.

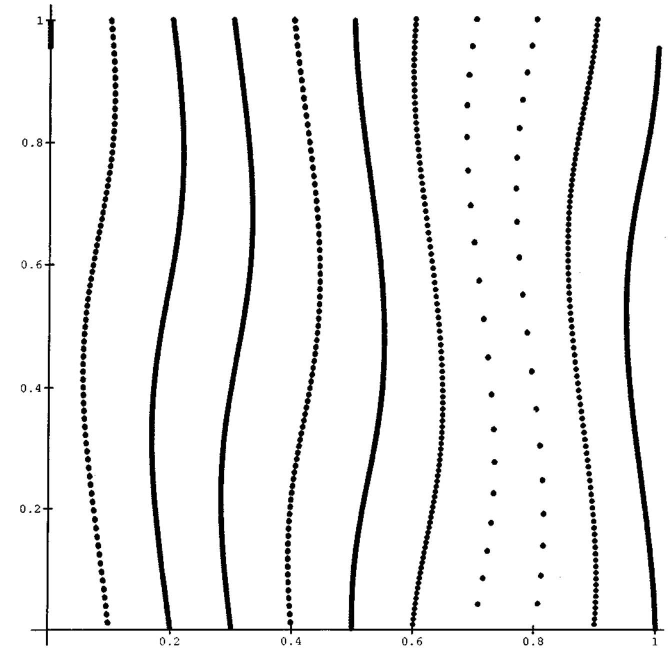

FIG. 10. The phase portrait of the isosceles triad with wave-vectors moduli $|\mathbf{k}_i|$ equal to 0.3, 1, 1, respectively. Here $\beta=2$.

oped [the ratio $\max(T_i)/T_M \sim \frac{1}{3}$ or greater] while the threshold for chaotic mixing corresponds to higher values of β.

A totally different bifurcation sequence exhibiting a standard scenario (cf., e.g., Ref. 1) of chaotic zones appearing at the separatrix crossing and then further developing via resonance overlap is observed for the isosceles triads, Figs. 10–12. As mentioned earlier, this is a degenerate case in a sense that modulation becomes harmonic. From the point of view of transport it is degenerate, too, as one of the wave vectors of an isosceles triad is necessarily vertical, which means that one of the triad's components is, in fact, a space-periodic zonal flow carrying Lagrangian particles to infinity.

V. DISCUSSION

Thus, we have shown that chaotic mixing accompanied in many cases by ballistic advection of a fraction of tracer particles characterize the Lagrangian transport by resonant triads of Rossby waves. However interesting by themselves, the question arises as to the relevance of these results for particle advection by the full flow where an initial wave triad excites, at a longer time scale, a broad spectrum of waves. It is true that the total vorticity conservation following from (1),

$$\frac{D\omega}{Dt}=0, \quad \omega=\Delta\psi+\beta y, \tag{26}$$

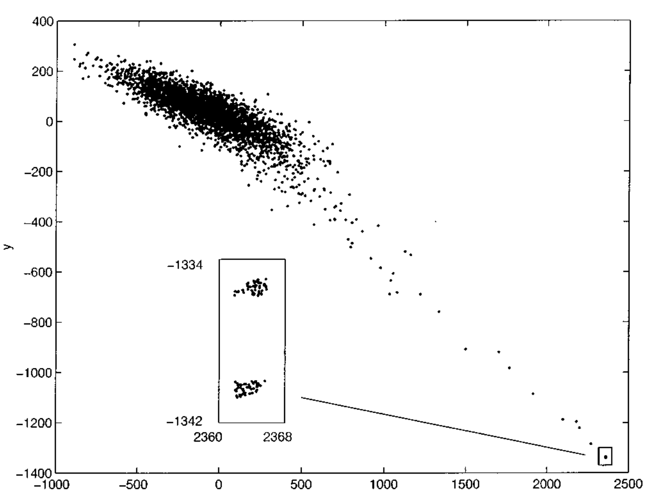

FIG. 9. The dispersion of tracer particles, initially uniformly distributed in the $(2\pi)^2$ box in the phase space after 500 modulation periods. The triad is the same as in Fig. 6.

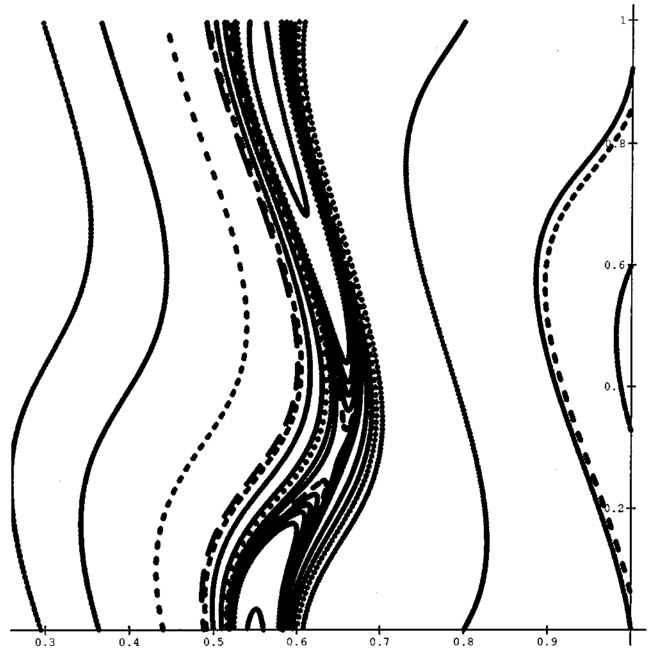

FIG. 11. The phase portrait of the isosceles triad with wave-vectors moduli $|\mathbf{k}_i|$ equal to 0.3, 1, 1, respectively. Here $\beta=1$.

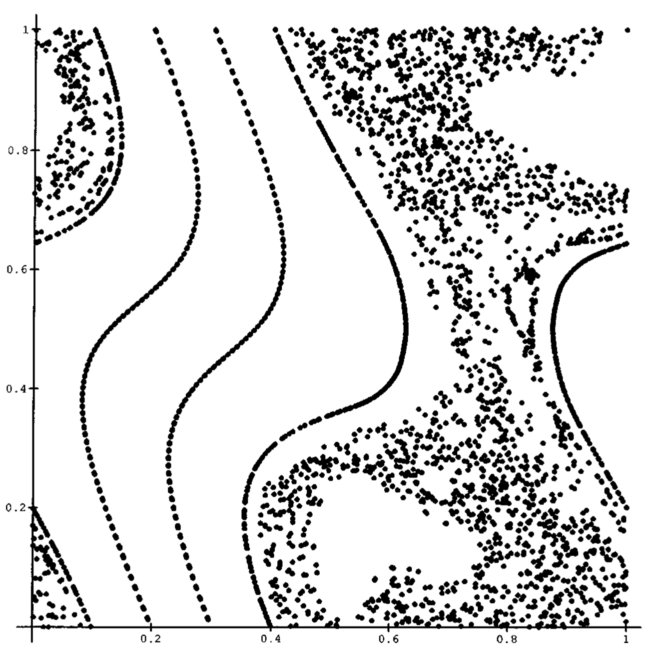

FIG. 12. The phase portrait of the isosceles triad with wave-vectors moduli $|\mathbf{k}_i|$ equal to 0.3, 1, 1, respectively. Here $\beta = 0.5$.

where D/Dt denotes the Lagrangian derivative, is not respected by (3), (6), which explains the fact that we observe chaotic behavior with a nonpiecewise constant total vorticity (cf. Ref. 8, where it was shown that a piecewise-constant profile of vorticity is a necessary condition for chaos in vorticity-conserving time-periodic flows). Another constraint following from (26) is that as total vorticity of any Lagrangian (=tracer) particle is constant its displacement in the y direction is limited from above by

$$y_2 - y_1 = \beta^{-1}(\Delta\psi_1 - \Delta\psi_2), \tag{27}$$

which implies the existence of an upper limit of possible dispersion in this direction determined by the magnitude of the variations of relative vorticity of the flow. Thus, the (nonhorizontal) ballistic advection at a very long time scale ($\gg \epsilon^{-1}$) in the full flow is questionable. Nevertheless, we believe that the ballistic transport we have found should contribute to the anomalous diffusion of tracers in the full flow. The fact that the nontrivial dynamical behavior we observed takes place in the domain of parameters corresponding to the transition from wave-dominated to vortex-dominated regimes in β-plane turbulence[21] suggests that the β-plane flows themselves may have interesting transport properties in this crossover regime. This remains to be checked in direct numerical simulations of β-plane turbulence. Note that in terms of the topology of the ω isolines this regime corresponds exactly to the appearance of closed contours (cf. Ref. 22). The role of the circulation cells in the anomalous diffusion in purely vortical two-dimensional turbulence is well known[23,24] (see Refs. 25 and references therein for a review of the anomalous diffusion problem). Much less is known on transport properties of more geophysically relevant systems, where waves and vortices may coexist, like β-plane and stratified turbulence. We should emphasize that in previous work the problem of diffusion by the ensembles of drift (=Rossby) waves was studied, neglecting the modulation of the wave amplitudes (cf. Refs. 4, 26). The transport by relaying triads of nonlinear Rossby waves found in the present study may provide a dynamical mechanism for anomalous diffusion. Note that if the role of the broad spectrum of waves generated at later times by the initial triad may be assimilated to turbulent diffusion the ballistic transport may persist in the resulting advection–diffusion equation according to the general analysis of Ref. 27.

Let us finally mention that the same mechanism exists in stratified turbulence where vortices coexist with internal gravity waves. In the Boussinesq approximation valid for short enough waves, where the effects of the wave-amplitude growth with height are neglected, the nonlinear gravity waves propagating in the vertical plane obey the following equations (the equations of the stratified turbulence in the vertical plane):

$$\begin{aligned} &\frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi) + \frac{\partial \xi}{\partial x} = 0, \\ &\frac{\partial \xi}{\partial t} + J(\psi, \xi) - N^2 \frac{\partial \psi}{\partial x} = 0, \end{aligned} \tag{28}$$

where ψ is the streamfunction of the velocity field in the vertical plane, ξ is the buoyancy variable, N is the Brunt–Väisäla frequency (see e.g., Ref. 22). The dispersion relation for the linear waves following from (28) is

$$\omega^2 = \Omega(k_x, k_y)^2 = N^2 \frac{k_x^2}{\mathbf{k}^2}, \tag{29}$$

and allows for resonant triads (see, e.g., Ref. 28 for the discussion of the role of resonant triads in ocean dynamics). Thus, an analysis similar to that made above for the Rossby waves may be done in this case, too, with similar conclusions.

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