Integrable four-dimensional symplectic maps of standard type

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Abstract

We search for rational, four-dimensional maps of standard type ($\mathbf{x}_{n+1} - 2\mathbf{x}_n + x_{n-1} = \varepsilon \mathbf{f}(\mathbf{x}, \varepsilon)$) possessing one or two polynomial integrals. There are no non-trivial maps corresponding to cubic oscillators, but we find a four-parameter family of such maps corresponding to quartic oscillators. This seems to be the only such example.

Suris [1] has found all one-degree-of-freedom Hamiltonian systems that possess *integrable* discretizations of standard type. If $f(x,0)$ is rational, it must be a polynomial of degree ≤ 3 and $f(x,\varepsilon)$ is rational. Naive counting suggests that if some 2–d.o.f. continuous systems possess a second integral, then some 2–d.o.f. symplectic maps might possess one integral. Here we search for such maps, which turn out to be rare: we find one four-parameter family of maps with one integral, a three-parameter subset of which is rotationally invariant and hence has a second integral; this subset corresponds to discretizations of the two-parameter family of Hamiltonians $\frac{1}{2}(p_1^2+p_2^2)+A(x_1^2+x_2^2)+B(x_1^2+x_2^2)^2$. The rarity is perhaps because we want a whole family of integrable maps (depending on ε), not just an isolated map.

A symplectic map of standard type is written

$$\mathbf{x}_{n+1}-2\mathbf{x}_n+\mathbf{x}_{n-1}=\varepsilon\mathbf{f}(\mathbf{x}_n,\varepsilon),\qquad \mathbf{f}(\mathbf{x},\varepsilon)=\nabla V(\mathbf{x},\varepsilon)\tag{1}$$

and the small parameter ε may be thought of as the square of the time-step in a discretization of $\ddot{\mathbf{x}}=\mathbf{f}(\mathbf{x},0)$. Assume that $\mathbf{f}$ and V have expansions

$$\mathbf{f}(\mathbf{x},\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j\mathbf{f}_j(\mathbf{x}),\qquad V(\mathbf{x},\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j V_j(\mathbf{x})$$

We want an integral $\Phi(\mathbf{x},\mathbf{y},\varepsilon)$ such that

$$\Phi(\mathbf{x}_{n+1},\mathbf{x}_n,\varepsilon)=\Phi(\mathbf{x}_n,\mathbf{x}_{n-1},\varepsilon)\qquad\text{for all } n\tag{2}$$

and assume that

$$\Phi(\mathbf{y},\mathbf{x},\varepsilon)=\Phi(\mathbf{x},\mathbf{y},\varepsilon)=\Phi_0(\mathbf{x},\mathbf{y})+\varepsilon\Phi_1(\mathbf{x},\mathbf{y}),$$

the first restriction following from the reversibility (under $n\mapsto -n$) of (1).

Our construction follows that of Suris. Define $\mathbf{u}_n=\mathbf{x}_n-\mathbf{x}_{n-1}+\varepsilon\mathbf{f}(\mathbf{x}_n,\varepsilon)/2$; then from (1) and (2) the condition for Φ to be an integral becomes

$$\Phi(\mathbf{x},\mathbf{x}+\varepsilon\mathbf{f}(\mathbf{x},\varepsilon)/2-\mathbf{u},\varepsilon)=\Phi(\mathbf{x},\mathbf{x}+\varepsilon\mathbf{f}(\mathbf{x},\varepsilon)/2+\mathbf{u},\varepsilon)\tag{3}$$

which may be expanded as a Taylor series in ε. At order ε^0, we get $\Phi_0(\mathbf{x},\mathbf{y})=\varphi(\mathbf{x}-\mathbf{y})$ (where φ is necessarily even), and at order ε,

$$(\nabla\varphi(\mathbf{u}))\cdot\mathbf{f}_0(\mathbf{x})=\Phi_1(\mathbf{x},\mathbf{x}-\mathbf{u})-\Phi_1(\mathbf{x},\mathbf{x}+\mathbf{u}).\tag{4}$$

In the one-degree-of-freedom case, Suris's method for solving the functional equation (3) is to:

(i) Differentiate the order ε term twice, the order ε^2 term once, and combine with the order ε^3 term; this gives six equations in the six unknowns $\Phi_1^{(i)}(x,x\pm u)$ $(i=1,2,3)$ which turn out to have rank 5. The consistency condition for these equations is separable in x and u, and he obtains $\varphi'''(u)/\varphi'(u)=c$, giving three *candidates* for φ: $u^2/2$, $(1-\cos\omega u)/\omega^2$, and $(\cosh\omega u-1)/\omega^2$.

(ii) For each candidate, deduce the functional form of Φ_1 from (4).

(iii) Take the most general function of this form, substitute into (3), and solve for $f(x,\varepsilon)$.

In each case, the candidate for φ did in fact give a family of solutions for f. With two degrees of freedom, the same steps could be followed: for (i), to get an overdetermined system it is necessary to differentiate the order ε term three times, etc., proceeding to the order ε^4 term; this gives a rank-18 set of twenty equations in twenty unknowns and two consistency conditions relating $\varphi^{(i,j)}(\mathbf{u})$ and $\mathbf{f}^{(i,j)}(\mathbf{x})$; these equations are complicated and probably not separable. However, the severest restriction is the new one that $\mathbf{f}(\mathbf{x}, \varepsilon)$ must be a gradient:

(iv) Given Φ_1, require $\frac{\partial f_1}{\partial x_2} \equiv \frac{\partial f_2}{\partial x_1}$.

This must be checked for any proposed φ. We have taken $\varphi(\mathbf{u}) = (u_1^2 + u_2^2)/2$ and searched for solutions of (3) satisfying (iv). From (4), $\Phi_1(\mathbf{x}, \mathbf{y})$ must be a polynomial, and at most quadratic in each variable, corresponding to a quartic potential V_0. Hence we take

$$\Phi_1(\mathbf{x}, \mathbf{y}) = \sum_{i,j,k,l=0}^{2} p_{ijkl} x_1^i x_2^j y_1^k y_2^l \qquad (p_{ijkl} = p_{klij} \quad \forall i, j, k, l).$$

The consistency conditions on $\mathbf{f}$ when solved directly from (3) are very complicated, so we determine them term-by-term. Once a solution at the first few orders is obtained we go back and check (3) directly. First determine V_0 from (4):

$$\begin{aligned} V_0 = -(&2\,x_2\,p_{0100} + x_2^2\,(p_{0101} + 2\,p_{0200}) + 2\,x_2^3\,p_{0201} + x_2^4\,p_{0202} + \\ &x_1\,\left(2\,p_{1000} + x_2\,(2\,p_{1001} + 2\,p_{1100}) + x_2^2\,(2\,p_{1002} + 2\,p_{1101}) + 2\,x_2^3\,p_{1102}\right) + \\ &x_1^2\,\left(p_{1010} + 2\,p_{2000} + x_2\,(2\,p_{1110} + 2\,p_{2001}) + x_2^2\,(p_{1111} + 2\,p_{2002})\right) + \\ &x_1^3\,(2\,p_{2010} + 2\,x_2\,p_{2011}) + x_1^4\,p_{2020}) \end{aligned}$$

and then at order ε^n $(n = 2, 3, 4)$ we have

$$\mathbf{u}.(\mathbf{f}_{n-1} + \mathbf{g}_n(\mathbf{x}).\mathbf{f}_{n-2} + \mathbf{h}_n(\mathbf{x})) = \mathbf{0} \tag{5}$$

where $\mathbf{g}_n(\mathbf{x})$ is a polynomial and $\mathbf{g}_2 = 0$. Solving (5) for $\mathbf{f}_{n-1}$ gives a consistency condition at that order and a value which can be substituted into the next order. The consistency conditions take the form of polynomials in x_1, x_2 (whose coefficients are functions of the p_{ijkl}) which must be zero; this gives polynomial equations, of degree n, in the p's. To simplify the equations we took various choices for V_0 and attempted to solve these consistency equations. (The actual equations and solutions are derived using a symbolic manipulator.) Without any other restrictions there are 14 equations at order 2, 27 at order 3, and 38 at order 4.

If the map is to be completely integrable, then the vector field it approximates as $\varepsilon \to 0$ must be also; hence we first tried V_0's corresponding to known integrable oscillators [2,3]: $Ax_1^2 + Bx_2^2 - x_1^2 x^2 - 2x_2^2$; $A(x_1^2 + 16x_2^2) - x_1^2 x_2 - 16x_2^3/3$ (two cases of the Hénon-Heiles Hamiltonian); $x_1^2 x_2 + 2x_2^3$; $x_1^4 + 12x_1^2 x_2^2 + 16x_2^2$; $x_1^4 + 6x_1^2 x_2^2 + 8x_2^2$; and $A(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2$. Only in the last case is there a consistent solution for the p's (given below), although sometimes it is necessary to continue to order ε^4 to reach the inconsistency.

Now consider whether there are any maps with only *one* integral. If so, then (4) shows (take $\partial/\partial\mathbf{u}$ and set $\mathbf{u} = 0$) that the integral must correspond to the Hamiltonian as $\varepsilon \to 0$, i.e. $\Phi_1(\mathbf{x}, \mathbf{x}) = -V_0(\mathbf{x})$. We are not interested in the trivial solutions—those which give linear maps, those which give uncoupled maps, or those which uncouple after a linear symplectic change of variables. For the form

$$V_0 = Ax_1^2 + Bx_2^2 + \alpha x_1^3 + \beta x_1^2 x^2 + \gamma x_1 x_2^2 + \delta x_2^3$$

the only such solutions found had $A = B$ and $3(\alpha\gamma + \beta\delta) = \beta^2 + \gamma^2$; a long calculation shows that these maps all uncouple after a linear symplectic change of variables. Thus there do not seem to be any nontrivial maps of this form with one integral corresponding to cubic oscillators.

For the form

$$V_0 = Ax_1^2 + Bx_2^2 + \alpha x_1^3 x_2 + \beta x_1 x_2^3$$

we found no nontrivial solutions.

For the form

$$V_0 = Ax_1^2 + Bx_2^2 + \alpha x_1^4 + \beta x_1^2 x_2^2 + \gamma x_2^4$$

we found a four-parameter family of nontrivial solutions with $\beta = 2\alpha = 2\gamma$ and arbitrary A, B. This seems to be the only nontrivial solution: Let

$$Q = b\left((1 - c\varepsilon)x_1^2 + (1 - d\varepsilon)x_2^2\right).$$

Then

$$\begin{gathered} V = -\frac{2 + a\varepsilon}{b\varepsilon^2}\ln(2(1 - c\varepsilon)(1 - d\varepsilon) - \varepsilon Q) - \frac{x_1^2 + x_2^2}{\varepsilon} \\ f_1 = x_1\frac{(a + 2d)(1 - c\varepsilon) + Q}{(1 - d\varepsilon)(1 - c\varepsilon) - \varepsilon Q/2}, \qquad f_2 = x_2\frac{(a + 2c)(1 - d\varepsilon) + Q}{(1 - d\varepsilon)(1 - c\varepsilon) - \varepsilon Q/2} \end{gathered} \tag{6}$$

with integral

$$-2\Phi_1 = a(x_1 y_1 + x_2 y_2) + b(x_1 y_1 + x_2 y_2)^2/2 + c(x_2^2 + y_2^2) + d(x_1^2 + y_1^2)$$

When $c = d$, the potential V is rotationally invariant and we expect a second integral; the method determines this automatically, for in this case one finds a term $e(x_1 y_2 - x_2 y_1)^2$ in Φ_1 with V independent of e. The obvious extension of this map to n degrees of freedom is also integrable:

$$\begin{aligned} \mathbf{f} &= \frac{a + 2c + b|\mathbf{x}|^2}{1 - c\varepsilon - \varepsilon b|\mathbf{x}|^2/2}\,\mathbf{x} \\ V &= -\frac{2 + a\varepsilon}{b\varepsilon^2}\ln\left(2 - b\varepsilon|\mathbf{x}|^2\right) - \frac{|\mathbf{x}|^2}{\varepsilon} \\ V_0 &= \frac{a + 2c}{2}|\mathbf{x}|^2 + \frac{b}{2}|\mathbf{x}|^4 \end{aligned} \tag{7}$$

Grammaticos et al. [4] have proposed an integrability test for maps: movable singularities should not propagate in time, and memory of the initial condition should survive the singularity. To apply the test (for simplicity, for 2 degrees of freedom), write the map in the form $\mathbf{x}_{n+1} = -x_{n-1} + \alpha/(1 - \beta^2|\mathbf{x}_n|^2)\mathbf{x}_n$ and take initial conditions $\mathbf{x}_0$ arbitrary, $\mathbf{x}_1 = (x_{11}, 0)$ (this suffices because we can rotate coordinates). Compute the next three iterates and then let $x_{11} \to 1/\beta$; this gives the iterates

$$(x_{00}, x_{01}), \quad (1/\beta, 0), \quad (\infty, -x_{01}), \quad (-1/\beta, 0), \quad (-x_{00}, x_{01})$$

showing that the proposed integrability test is satisfied here.

We can now use the new "angular momentum" integrals to reduce the map to one degree of freedom. This illustrates that the two processes, (i) forming an integrable map approximating a continuous system and (ii) reducing by the rotational symmetry, do not commute. Rotate coordinates so that $x_i = y_i = 0$ for $i > 2$. Write the map as

$$\begin{aligned}\mathbf{x}' &= \mathbf{x} + \mathbf{p}\\ \mathbf{p}' &= \mathbf{p} + \mathbf{x}' g(|\mathbf{x}'|)\end{aligned}$$

and define new variables (chosen to correspond to the continuous case)

$$\begin{aligned}L &= x_1 p_2 - x_2 p_1\\ (r,\theta): &\quad \text{polar coordinates for } (x_1, x_2)\\ p_r &= (x_1 p_1 + x_2 p_2)/r\\ p_\theta &= L\end{aligned}$$

in which the map can be written (after much algebra)

$$\begin{aligned}(r^2)' &= \frac{L^2}{r^2} + (p_r + r)^2\\ (r p_r)' &= \frac{L^2}{r^2} + r p_r + p_r^2 + (r')^2 g(r')\\ &= (r')^2(1 + g(r')) - r(r + p_r)\\ L' &= L\\ \theta' &= \theta + \sin^{-1}\frac{L}{r r'}\end{aligned} \tag{8}$$

with integral

$$\frac{L^2}{r^2} + p_r^2 - \varepsilon\left(r(r+p_r)\,(a + br(r+p_r)/2) + c\left(\frac{L^2}{r^2} + (r+p_r)^2 + r^2\right)\right)$$

However, starting with the continuous system corresponding to (7) ($\ddot{\mathbf{x}} = \nabla V_0(|\mathbf{x}|)$) and performing the corresponding reduction gives

$$\ddot{r} = \frac{L^2}{r^3} + V_0'(r), \qquad \dot{\theta} = \frac{L}{r^2}$$

which does *not* have an integral discretization of standard type. So we have extended Suris's list of systems possessing integrable discretizations by one; the catch is the our reduced system (8) is not of standard type. This suggests that there may be rational integrable discretizations of any rational Hamiltonian if one enlarges the allowed class of maps.

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References

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