A NOTE ON THE MOTION OF SURFACES

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ABSTRACT. We study the motion of surfaces in an intrinsic formulation in which the surface is described by its metric and curvature tensors. The evolution equations for the six quantities contained in these tensors are reduced in number in two cases: (i) for arbitrary surfaces, we use principal coordinates to obtain two equations for the two principal curvatures, highlighting the similarity with the equations of motion of a plane curve; and (ii) for surfaces with spatially constant negative curvature, we use parameterization by Tchebyshev nets to reduce to a single evolution equation. We also obtain necessary and sufficient conditions for a surface to maintain spatially constant negative curvature as it moves. One choice for the surface's normal motion leads to the modified-Korteweg de Vries equation, the appearance of which is explained by connections to the AKNS hierarchy and the motion of space curves.

1. INTRODUCTION AND EQUATIONS OF MOTION

Studies of the motion of surfaces and interfaces occur in several scientific disciplines, including geometry [8,10,14,18,19], water waves [22], crystal growth [11,13], combustion [4], and gas dynamics [20]. Pelcé [17] collected and reprinted an interesting collection of original articles in some of these areas. To simplify the analysis, these studies often consider only two-dimensional motion, thus reducing the problem to the motion of a curve [6], but this is only a special case of the usual situation.

To study the evolution of two-dimensional surfaces arbitrarily embedded in three dimensions, a natural approach is to describe the surface purely intrinsically, that is, in terms of the surface's metric and curvature tensors. In these variables, the equations of motion for the surface take the form of six coupled nonlinear evolution equations for the six independent components of the two tensors. These equations were first written down by Brower et al. [3], and later by Nakayama & Wadati [16]. But these equations are highly redundant: the surface has codimension one and should be described by only one independent function.

The best formulation of this problem is not yet known. Here we take a cue from the successful treatment of the motion of curves in terms of local curvature, and obtain evolution equations for surfaces in terms of similar intrinsic quantities. We simplify the evolution equations of [3] and [16] by reducing the number of unknowns, obtaining two simplifications by choosing two different parameterizations of the surface. In both cases, the key idea is to force the coordinate

lines to maintain their defining property as the surface evolves, by adding an appropriate tangential velocity to the evolution.

The first parameterization, in §2, is general: we parameterize a surface by its principal lines (i.e., lines tangent to the directions of principal curvature). This reduces the system to four coupled equations, and, with a particular choice of parameterization along the principal lines, to two equations for the two curvatures. If the surface happens to be a cylinder above a curve in a plane, and if the motion is two-dimensional, then one curvature vanishes and the equations reduce to the single equation obtained in [3,15] for the evolution of the curvature of a plane curve.

The second parameterization, in §3, is more specialized: surfaces of constant negative curvature have an elegant parameterization by Tchebyshev nets. For surfaces that maintain constant negative curvature as they move, their motion is specified by a single evolution equation for the angle between the coordinate lines. As an example, we show that one choice for the normal velocity of the surface leads to the modified Korteweg-de Vries equation for the surface's evolution.

The first step is to derive the six coupled equations for the motion of a surface moving in $\mathbf{R}^3$. Local coordinates on the surface are $(x_1, x_2) = \mathbf{x}$, time is t, and the position (in $\mathbf{R}^3$) of a point on the surface is $\mathbf{r}(\mathbf{x}, t)$. Later $\dot{\mathbf{r}}$ ($= \partial \mathbf{r}/\partial t$) will be specified, so one can think of $\mathbf{x}$ as a Lagrangian particle label on the surface. Define two tangent vectors and one normal vector:

$$\boldsymbol{\tau}_\mu = \mathbf{r},_\mu \qquad \mathbf{n} = \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2|},$$

where greek indices range over $\mu = 1, 2$ with $,_\mu = \frac{\partial}{\partial x_\mu}$. (Both the notation used here and the preliminary development follow Spivak [21, Chapter 2].) Note that $\mathbf{n}$ is a unit vector but the $\boldsymbol{\tau}_\mu$ are not necessarily unit vectors. We have the metric $g_{\mu\nu}$ and curvature $h_{\mu\nu}$ tensors (which define the first and second fundamental forms $g_{\mu\nu}dx_\mu dx_\nu$ and $h_{\mu\nu}dx_\mu dx_\nu$, respectively):

$$g_{\mu\nu} = \boldsymbol{\tau}_\mu \cdot \boldsymbol{\tau}_\nu, \qquad h_{\mu\nu} = \mathbf{n} \cdot \boldsymbol{\tau}_{\mu,\nu} = \mathbf{n} \cdot \mathbf{r},_{\mu\nu} \qquad (1.1)$$

As one moves along the surface (at a fixed time), the tangent and normal vectors change according to the Gauss-Weingarten equations,

$$\begin{aligned} \boldsymbol{\tau}_{\mu,\nu} &= \boldsymbol{\tau}_\lambda \Gamma^\lambda{}_{\mu\nu} + \mathbf{n} h_{\mu\nu} \\ \mathbf{n},_\nu &= -\boldsymbol{\tau}_\lambda g^{\lambda\mu} h_{\mu\nu} \end{aligned} \qquad (1.2)$$

where $\Gamma^\lambda{}_{\mu\nu}$ are the Christoffel symbols of the second kind defined by the metric $g_{\mu\nu}$, and $g^{\mu\nu} = (g_{\mu\nu})^{-1}$ as usual. Repeated indices are summed on unless otherwise noted.

The six quantities in (1.1) are not all independent. They are related by three consistency conditions for the PDE's (1.2)—the Gauss-Codazzi equations:

$$R_{1212} = \det h \qquad (1.3a)$$

$$h_{\nu\lambda;\mu} = h_{\mu\lambda;\nu} \qquad (1.3b)$$

where $;_\mu$ denotes covariant derivative and R_{1212} is the nontrivial element of the Riemann tensor. Note that (1.3b) gives independent information only for

$(\nu, \lambda, \mu) = (1, 1, 2)$ or $(2, 2, 1)$. Of the three remaining degrees of freedom, two are due to the arbitrary parameterization, leaving one—which we expect, since the surface could be written, e.g., $\mathbf{r} = (\mathbf{x}, z(\mathbf{x}), t)$.

Therefore one might ask: What is to be gained by going to an "intrinsic" (i.e., $g_{\mu\nu}$, $h_{\mu\nu}$) representation of the surface when eqs. (1.3) cannot usually be solved explicitly? Any surface motion can after all be formulated locally as $\dot{z} = f(\mathbf{x}, t)$. Such an approach fails if the surface becomes vertical—an apparently artificial constraint—but the intrinsic formulation can also suffer from artificial coordinate singularities. One advantage is that in some applications, the motion of the surface normal to itself is given naturally in terms of the local curvatures [3,19] which are the eigenvalues of $g^{\mu\lambda}h_{\lambda\nu}$; secondly, such an approach has been successful in the study of the motion of curves, as we now discuss.

Hasimoto [9] showed that in the self-induction approximation, the motion of a 3D vortex filament (which moves in the direction of its local binormal vector with speed equal to its local curvature) can be reduced to the nonlinear Schrödinger equation. Lamb [12] extended this to more general motions of curves, and found motions obeying the sine-Gordon and modified Korteweg-de Vries equations. All three of these equations are completely integrable [1], and Goldstein and Petrich [7] showed how to specify infinitely many two-dimensional motions of curves in terms of integrable evolution equations. The occurrence of such integrable equations was shown in [15] to be due to the fact that the Serret-Frenet equations, $(\mathbf{t} = \boldsymbol{\tau}/|\boldsymbol{\tau}|, s = \text{arclength})$

$$\begin{pmatrix} \mathbf{t}_{,s} \\ \mathbf{n}_{,s} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} \tag{1.4}$$

which are the analog of (1.2) for plane curves, are equivalent to the AKNS scattering problem [1] at zero eigenvalue. Thus evolutions $\begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \end{pmatrix}$ can be specified which give rise to integrable equations of the AKNS hierarchy.

The equations for surfaces, (1.2), are more complicated than those for curves, (1.4), for two main reasons: (i) $\mathbf{t}$ in (1.33) is a unit vector, whereas the $\boldsymbol{\tau}_\mu$ in (1.2) are not; and (ii) the coordinates, x_i, in (1.2) need not measure arclength. If we relax these two requirements in (1.4), then they become

$$\begin{pmatrix} \boldsymbol{\tau}_{,x} \\ \mathbf{n}_{,x} \end{pmatrix} = \begin{pmatrix} g_{,x}/2g & h \\ -h/g & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{n} \end{pmatrix}$$

which are equivalent to (1.2) when $\mu = 1$ only.

There is another interesting connection between curves and surfaces which we note briefly. In [15] it is shown that the Serret-Frenet equations for *space* curves also reduce to the AKNS scattering problem at zero eigenvalue (i.e., they take the form (1.4) in appropriate variables). Considering a moving space curve as sweeping out a surface, we see that the equations analogous to Serret-Frenet for the surface, the Gauss-Weingarten equations (1.2), must also contain the AKNS scattering problem (in appropriate variables). This point of view explains the occurrence of integrable equations from the AKNS hierarchy in the equations defining particular surfaces, such as the sine-Gordon equation for surfaces of constant negative curvature.

Motion of a surface can be prescribed in the form

$$\dot{\mathbf{r}} = U\mathbf{n} + V^\mu \boldsymbol{\tau}_\mu \tag{1.5}$$

where U and V^μ can be specified arbitrarily. This implies (differentiate $\boldsymbol{\tau}_\mu = \mathbf{r}_{,\mu}$ and $\mathbf{n} \cdot \boldsymbol{\tau}_\mu = 0$ and use (1.2)):

$$\begin{aligned}
\dot{\boldsymbol{\tau}}_\mu &= (U_{,\mu} + V^\nu h_{\mu\nu})\mathbf{n} + (-Uh^\lambda{}_\mu + V^\lambda{}_{;\mu})\boldsymbol{\tau}_\lambda \\
\dot{\mathbf{n}} &= -g^{\lambda\mu}(U_{,\mu} + V^\nu h_{\mu\nu})\boldsymbol{\tau}_\lambda
\end{aligned} \tag{1.6}$$

Now intrinsic evolution equations for $\dot{g}_{\mu\nu}$, $\dot{h}_{\mu\nu}$ can be derived in two ways. The first is to differentiate (1.1) with respect to t and to eliminate $\boldsymbol{\tau}_\mu$ and $\mathbf{n}$ using (1.2, 1.3). This was first done by Brower [3] for $V^\mu = 0$ and by Nakayama et al. [16] for $V^\mu \neq 0$. There is also an "intrinsic" derivation, namely requiring compatibility of the PDE's (1.2) and (1.6). This gives equations which must be solved for $\dot{g}_{\mu\nu}$ and $\dot{h}_{\mu\nu}$, and the result is (of course) the same:

$$\begin{aligned}
\dot{g}_{\mu\nu} &= -2h_{\mu\nu}U + V_{\mu;\nu} + V_{\nu;\mu} \\
\dot{h}_{\mu\nu} &= U_{;\mu\nu} - h_{\mu\lambda}h^\lambda{}_\nu U + h_{\mu\lambda}V^\lambda{}_{;\nu} + h_{\nu\lambda}V^\lambda{}_{;\mu} + V^\lambda h_{\mu\lambda;\nu}
\end{aligned} \tag{1.7}$$

Notice that V^μ does not affect the actual shape of the surface. The easiest way to see this is to note that the evolution in (1.5) is linear in U, V^μ, so we can consider $U = 0$; then the V^μ velocities just push particles around on a motionless surface. Notice also that if a coordinate system with some particular defining property is chosen initially, that property will be destroyed under general motion of the particles on the surface. Therefore we will use the tangential velocities V^μ to "push" the coordinates back, so that they maintain their defining property as the surface moves. (In the next sections we show how to do this in some special cases.) Nakayama et al. [16] in addition allowed the coordinates to evolve on the surface, but this does not give any extra generality.

We also direct the reader to the detailed work of Pinkall and Sterling [19] on constant mean curvature surfaces. They use principal coordinates and apply a motion, as above, with V^μ chosen to maintain constant mean curvature, and, further, obtain families of such surfaces.

2. PRINCIPAL COORDINATES

The principal curvatures (κ_1, κ_2) and principal directions of a surface are the eigenvalues and eigenvectors of $g^{\mu\lambda}h_{\lambda\nu}$. If the coordinate lines are chosen to be tangent to the principal directions at each point on the surface, then $g_{\mu\nu}$ and $h_{\mu\nu}$ are diagonal, and we have *principal coordinates*. These can be defined globally on any surface, and are smooth except where $\kappa_1 = \kappa_2$—umbilic points. (See [21, p. 288] for more information about umbilic points; they are generally isolated.) We define g_1, g_2, κ_1, and κ_2 by

$$g_{\mu\nu} = \begin{pmatrix} e^{g_1} & 0 \\ 0 & e^{g_2} \end{pmatrix}, \qquad h_{\mu\nu} = \begin{pmatrix} e^{g_1}\kappa_1 & 0 \\ 0 & e^{g_2}\kappa_2 \end{pmatrix}.$$

Note that g_i and κ_i do not form covariant vectors—they are just functions appearing in the 2-tensors $g_{\mu\nu}$ and $h_{\mu\nu}$.

In terms of these new variables, the Gauss-Codazzi equations (1.3b) now read

$$g_{1,2} = \frac{2\kappa_{1,2}}{\kappa_2 - \kappa_1}, \qquad g_{2,1} = -\frac{2\kappa_{2,1}}{\kappa_2 - \kappa_1} \tag{2.1}$$

which we will use to simplify the subsequent equations. For completeness, we also give the Gauss-Weingarten formulae in these variables:

$$\frac{\partial}{\partial x_1} \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}g_{1,1} & -\frac{1}{2}g_{1,2}e^{g_1 - g_2} & e^{g_1}\kappa_1 \\ \frac{1}{2}g_{1,2} & \frac{1}{2}g_{2,1} & 0 \\ -\kappa_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \mathbf{n} \end{pmatrix}$$

$$\frac{\partial}{\partial x_2} \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}g_{1,2} & \frac{1}{2}g_{2,1} & 0 \\ -\frac{1}{2}g_{2,1}e^{g_2 - g_1} & \frac{1}{2}g_{2,2} & e^{g_2}\kappa_2 \\ 0 & -\kappa_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \mathbf{n} \end{pmatrix} \tag{2.2}$$

As the surface moves, the coordinate lines will not remain principal coordinates unless $\dot{g}_{12} = \dot{h}_{12} = 0$. (This is where we begin to specify V^μ in order to maintain the desired coordinate system.) This leads to two equations in $V^1_{,2}$ and $V^2_{,1}$ whose solution is

$$V^1_{,2} = e^{-g_1}f, \qquad V^2_{,1} = -e^{-g_2}f \tag{2.3}$$

where

$$f = \frac{(U_{,2}\kappa_2)_{,1} - (U_{,1}\kappa_1)_{,2}}{(\kappa_2 - \kappa_1)^2}.$$

With these choices, (1.7) reduces to four coupled equations:

$$\dot{g}_i = -2\kappa_i U + V^j g_{i,j} + 2V^i_{,i} \tag{2.4a}$$

$$\dot{\kappa}_i = \kappa_i^2 U + V^j \kappa_{i,j} + e^{-g_i/2}(e^{-g_i/2}U_{,i})_{,i} + e^{-g_{i'}}g_{i,i'}U_{,i'}/2 \tag{2.4b}$$

$((i, i') = (1, 2)$ or $(2, 1)$; no sum on i, $i')$. These may be written as

$$\frac{Dg_i}{Dt} - 2V^i_{,i} = -2\kappa_i U$$
$$\frac{D\kappa_i}{Dt} = \kappa_i^2 U + U_{,s_i s_i} + e^{-g_{i'}}g_{i,i'}U_{,i'}/2 \tag{2.5}$$

where D/Dt is total derivative in the direction $-V^j\partial_j$ (i.e., the right hand side of (2.5) gives the evolution at "unpushed" particle labels) and s_i is arclength along x_i, (i.e., $s_1 = \int^{x_1} \exp(\frac{1}{2}g_1(\acute{x}_1, x_2)) \, d\acute{x}_1$).

As a special case of these equations, we note that if $\kappa_2 = \partial_2 = 0$ so that the surface only depends on x_1, then $f = 0$ in (2.3), we can take $V^2 = 0$, $V^1 = W(x_1)\exp(-\frac{1}{2}g_1)$ say, and we recover the equations for the motion of a plane curve [15]:

$$\dot{g} = 2W_{,s} - 2\kappa U \qquad \dot{\kappa} = \kappa^2 U + U_{,ss} + W\kappa_{,s}$$

Now we return to the general case, to reduce (2.4) to two coupled equations for the evolution of the two curvatures, κ_1 and κ_2. These two equations, along

with their compatibility condition, (1.3a), seem to be the simplest intrinsic formulation possible for a general surface.

In (2.3), the equations for V^μ, there is still the freedom to specify two arbitrary functions, each of one variable. We can use this freedom to specify g_i, and hence the metric, completely. Here is one way to specify the metric: require

$$g_1(x_1, 0, t) = g_2(0, x_2, t) = 0 \quad \forall\, t, \tag{2.6}$$

so that $g_{11} = g_{22} = 1$ along these two coordinate lines, which are both parameterized by arclength (see Figure 1). Let v^μ be two functions of integration in (2.3), i.e.,

$$V^1(x_1, x_2) = + \int_0^{x_2} \exp(-g_1(x_1, \acute{x}_2)) f(x_1, \acute{x}_2)\, d\acute{x}_2 + v^1(x_1)$$

$$V^2(x_1, x_2) = - \int_0^{x_1} \exp(-g_2(\acute{x}_1, x_2)) f(\acute{x}_1, x_2)\, d\acute{x}_1 + v^2(x_2)$$

where we have suppressed the explicit t dependence. Then, from (2.4a) and choosing $v_i(0) = 0$, enforcing (2.6) requires

$$v_i' = (\kappa_i U - \tfrac{1}{2} g_{i,i'} V^{i'})\big|_{x_{i'}=0}$$

Now (2.1) can be integrated to define g_i completely:

$$g_1 = \int_0^{x_2} \frac{2\kappa_{1,2}}{\kappa_2 - \kappa_1}\, d\acute{x}_2, \qquad g_2 = -\int_0^{x_1} \frac{2\kappa_{2,1}}{\kappa_2 - \kappa_1}\, d\acute{x}_1. \tag{2.7}$$

Substituting these into (2.4b) yields two coupled equations for the evolution of the two curvatures, as desired.

Thus, we have in (2.4b), (2.7), the evolution equations for the principal curvatures that are most closely analogous to the well-known evolution equations for the curvature and torsion of a space curve [7]. For a (moving) surface, the principal curvatures are not independent; they are still constrained by Gauss's equation (1.3a), but this constraint cannot be eliminated easily in this representation. Even so, these two coupled evolution equations for the principal curvatures are certainly more compact than the (equivalent) six coupled equations in [3] and [16]. This simplification is the main result in this section.

In the $\kappa_2 = \partial_2 = 0$ case, $v_1' = \kappa_1 U$ and in this gauge, x_1 is arclength, so we get $W = \int \kappa_1 U\, ds$ as expected.

3. Surfaces of constant negative curvature

From the wealth of special surfaces and coordinate systems available, we consider a particular choice for which the equations of motion take a particularly simple form. At every point on a surface whose Gaussian (or total) curvature K is negative, the curvature tensor has two null vectors (i.e., vectors v^μ which annihilate $h_{\mu\nu}$, so $v^\mu v^\nu h_{\mu\nu} = 0$; see [21, p. 70]). An *asymptotic line* is a smooth curve on the surface whose tangent vector coincides with one of these null vectors at each point along the curve. If the Gaussian curvature is constant (in space) and negative, then there are two sets of asymptotic lines, and they form

a *Tchebyshev net*, i.e., they can be parameterized by arclength everywhere [21, p. 368]. With these as coordinate lines, the metric and curvature tensors take the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}, \qquad h_{\mu\nu} = \begin{pmatrix} 0 & h_{12} \\ h_{12} & 0 \end{pmatrix}, \qquad (3.1)$$

where ω measures the angle between the coordinate lines. Furthermore, the Gauss-Codazzi equations become:

$$h_{12} = \sqrt{-K}\sin\omega, \qquad (3.2a)$$

$$\omega_{,12} = -K\sin\omega, \qquad (3.2b)$$

i.e., solutions of the sine-Gordon equation are in a local 1–1 correspondence with such surfaces. Unfortunately such coordinates may be only local [21, p. 373]. The Gauss-Weingarten equations (1.2) also take the simple form

$$T_{,1} = \begin{pmatrix} \omega_{,1}\cot\omega & -\omega_{,1}\csc\omega & 0 \\ 0 & 0 & \sqrt{-K}\sin\omega \\ \sqrt{-K}\cot\omega & -\sqrt{-K}\csc\omega & 0 \end{pmatrix} T \qquad (3.3)$$

$$= \begin{pmatrix} -(\ln(\kappa_1 - \kappa_2))_{,1} & -(\ln(\kappa_1 + \kappa_2))_{,1} & 0 \\ 0 & 0 & -\kappa_1\kappa_2/(\kappa_1 - \kappa_2) \\ \kappa_1 + \kappa_2 & -(\kappa_1 - \kappa_2) & 0 \end{pmatrix} T$$

and similarly for $T_{,2}$, where

$$T = \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \\ \mathbf{n} \end{pmatrix}$$

and κ_1, κ_2 are the principal curvatures $\sqrt{-K}(\cot\omega \pm \csc\omega)$.

The first step is to find those normal velocities U that keep the Gaussian curvature constant in space (but not necessarily in time). Substitute (3.1) into (1.7), and require both that the form of (3.1) be preserved (i.e., $\dot{g}_{11} = \dot{g}_{22} = \dot{h}_{11} = \dot{h}_{22} = 0$) and consistency (so that $\dot{\omega}$ from (1.7a) equals that from (1.7b)). One finds in this way that U must satisfy a linear inhomogeneous equation:

$$U_{,12} = -KU\cos\omega + (\sin\omega)\frac{d}{dt}\sqrt{-K} \qquad (3.4)$$

and that the tangential components of velocity must satisfy:

$$V^i_{,i'} = -(KU\sin\omega + \omega_{,i'}((-1)^{i'}U_{,i'}\cos\omega + U_{,i}) + U_{,i'i'}\sin\omega)/2\sqrt{-K}\sin^2\omega \quad (3.5)$$

where i, $i' = (1,2)$ or $(2,1)$, and there is no sum on i'. There are no other constraints. As mentioned earlier, only the normal component of velocity actually moves the surface, so we have established the following

Proposition: *Suppose that a surface whose Gaussian curvature K is constant in space and negative moves according to (1.5). It will maintain (spatially) constant negative curvature if and only if U satisfies (3.4).*

Note that if $\dot{K} = 0$, then (3.4) is just the linearization of the sine-Gordon equation, (3.2b), so its translational symmetry immediately gives two solutions

of (3.4): $U = \omega_{,1}$ and $U = \omega_{,2}$. With the velocity components chosen to satisfy (3.4) and (3.5), the evolution of ω is given by

$$\frac{D\omega}{Dt} + (V^1_{,2} + V^2_{,1})\sin\omega = 2\sqrt{-K}U \qquad (3.6)$$

where D/Dt was defined below (2.5).

As an example, take $\dot{K} = 0$ and $U = \omega_{,1}$ ($= \omega_{,s_1}$). Eqs. (3.5) can be integrated to find

$$V^1 = ((\omega_{,1})^2 + 2\omega_{,11}\cot\omega)/4\sqrt{-K}, \qquad V^2 = -\omega_{,11}\csc\omega/2\sqrt{-K}$$

so

$$\dot{\omega} = \frac{1}{4}\left(6\sqrt{-K}\omega_{,1} + ((\omega_{,1})^3 + 2\omega_{,111})/\sqrt{-K}\right) \qquad (3.7)$$

which is the modified Korteweg-de Vries equation in the variable $\omega_{,1}$. The occurrence of an equation from the AKNS hierarchy is not surprising, for the following reason. Let $\omega(x_1, 0)$ be an arbitrary function. It can be integrated in x_2 under (3.2b) to sweep out a surface of constant negative curvature, or (because (3.7) does not depend explicitly on x_2) it can be integrated in t under (3.7). These two AKNS evolution equations commute—thus (3.7) obviously preserves the constant curvature property. In a sense, one should regard both x_2 and t as time-like variables. A similar instance of time-like evolution in spatial directions was reported in [2]. It is not known whether more complicated evolutions (e.g. $U = \omega_{,1} + \omega_{,2}$) can give rise to integrable equations.

Now that it is known that this surface motion can be expressed as the motion of a space curve it is fruitful to explore further. The coordinate lines are asymptotic and hence their principal unit normals lie in the surface (this can be seen from (3.3), because $T_{1,1}$ is proportional to the principal unit normal of the x_1-curve). Furthermore for the particles in a coordinate line to trace out the coordinate grid properly it is easy to see (see Figure 2) that

$$\mathbf{r}_{,2} = \sin\omega\,\mathbf{n}_c + \cos\omega\,\mathbf{t}_c \equiv U_c\mathbf{n}_c + W_c\mathbf{t}_c$$

where the subscript $_c$ denotes properties of an x_1-coordinate curve, κ_c is the curvature of the coordinate curve, and τ_c is its torsion. We know that this "motion" (i.e., motion in the x_2-direction, as the curve sweeps out the initial surface) maintains the arc-length parameterization, which requires $W_{,1} = \kappa_c U \Rightarrow \kappa_c = -\omega_{,1}$. Now take this curve motion and find $\kappa_{c,2}$ using the space-curve evolution equations of [15]: the result is

$$\kappa_{c,2} = -\tau^2\sin\omega.$$

Because ω satisfies the sine-Gordon equation, we get $\tau = \sqrt{-K}$, i.e., Tchebyshev coordinate lines in a surface of constant negative curvature have constant torsion [5]. The curve is evolving under the equation of motion

$$\mathbf{r}_{,2} = -\sin\int\kappa_c\,ds\,\mathbf{n}_c + \cos\int\kappa_c\,ds\,\mathbf{t}_c \qquad (3.8)$$

We have in fact recovered the constant-torsion sine-Gordon curve equation of Lamb [12]. However, due to his inverse method (specifying $\dot{\mathbf{t}}$, $\dot{\mathbf{n}}$, $\dot{\mathbf{b}}$ instead of $\dot{\mathbf{r}}$),

he was unable to determine $\dot{\mathbf{r}}$ for this evolution. In the forward direction it is easy to check that we have in fact the same evolution.

(NOTE: In the literature there are two distinct curve motions which obey the sine-Gordon equation. Let the curvature evolve by $\dot{\kappa}_c \equiv \Omega U$. Lamb's evolution, given by (3.8), has $\Omega U = 0$, and the curve would not move at all if $\tau_c = 0$. Nakayama et al. [15] found a *planar* curve motion which obeys sine-Gordon: it has $\Omega^2 U = 0$.)

As pointed out above, we can also think of an x_1-coordinate line as moving in the t-direction, governed by the equation of motion (see Figure 2)

$$
\begin{aligned}
\dot{\mathbf{r}} &= U\mathbf{b}_c + V^1\mathbf{t}_c + V^2(\cos\omega\,\mathbf{t}_c + \sin\omega\,\mathbf{n}_c) \\
&= -\kappa_c\mathbf{b}_c + \kappa_{c,1}\mathbf{n}_c/2\sqrt{-K} + \kappa_c^2\mathbf{t}_c/4\sqrt{-K}
\end{aligned}
\tag{3.10}
$$

The final cross-check is that a space curve with motion given by (3.10) has (from the curve equations in Nakayama et al. [15]) intrinsic evolution

$$
\dot{\kappa}_c = \frac{3}{2}\sqrt{-K}\kappa_{c,s} + (3\kappa_c^2\kappa_{c,s} + 2\kappa_{c,sss})/4\sqrt{-K}
$$
$$
\dot{\tau}_c = 0
$$

agreeing with (3.7) when $\kappa_c = -\omega_{,1}$.

In summary, for a (moving) surface of constant negative curvature, the intrinsic formulation together with specially enforced coordinate systems interrelates naturally with the dynamics of curves and the surfaces they sweep out. The two points of view provide a fruitful alternative to the traditional representations of surface motion.

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